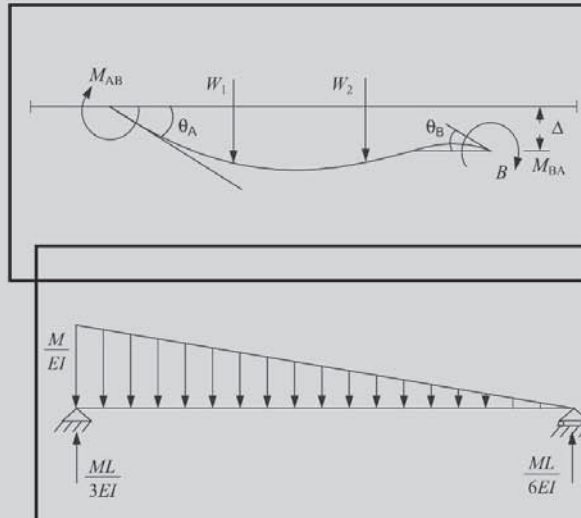


Slope Deflection Method



Chapter Outline

- 1.1 Introduction
- 1.2 Assumptions
- 1.3 Sign Conventions
- 1.4 Derivation of Slope Deflection Equations
- 1.5 Applications of Slope Deflection Equations

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1.1 INTRODUCTION

The credit of giving a final touch to slope deflection method and publishing it (1915) goes to *Prof. G A Maney* of the University of Minnesota. This method is ideally suited for the analysis of continuous beams and rigid jointed frames. Using this method basic unknowns like slopes and deflections of joints can be calculated. Moments at the ends of a member is first written in terms of unknown slopes and deflections of the end joints. Considering the joint equilibrium conditions, a set of equations are formed and solution of these simultaneous equations gives unknown slopes and deflections. Then end moments of individual members are determined. Since, it involves solution of simultaneous equations, a problem with more than three unknowns was considered as difficult problem for hand calculation. Hence, this method was sidelined by moment distribution method. With the help of computers, solutions for any number of simultaneous equations can be obtained easily. The development of this method in the matrix form, has led to the '**Stiffness Matrix Method**,' which is commonly used for the analysis of large structures with the help of computers. Hence, study of this method has gained importance again.

1.2 ASSUMPTIONS

The following assumptions are made while developing this method:

1. All joints are rigid, *i.e.*, the angle between any two members in a joint does not change even after deformation due to loading. Thus, at joint *A* in the Figure 1.1, the angle between members *AB* and *BC* remains ' θ ' only, even after deformation.

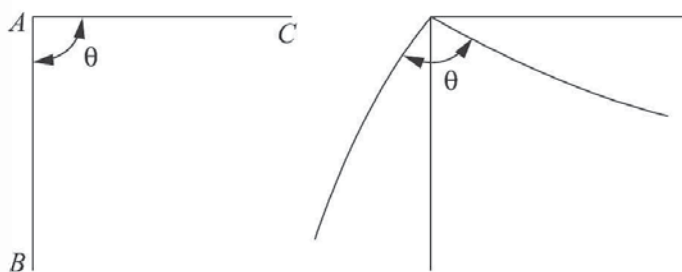


Figure 1.1(a): Rigid joint before deformation. **(b)** Rigid joint after deformation.

2. Distortions due to axial deformations are neglected. Thus, in the frame shown in Figure 1.2,

$$BB' = CC' = \Delta$$

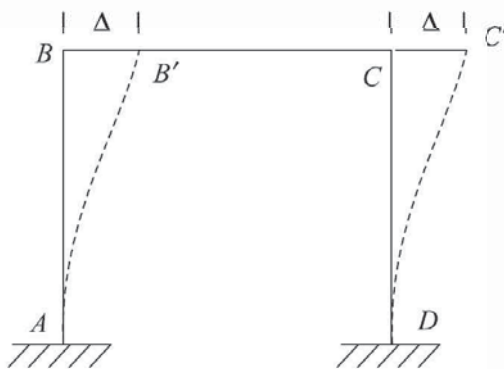


Figure 1.2: Axial deformation neglected.

3. Shear deformations are neglected.

Assumptions 2 and 3 are reasonable because, distortion of rigid jointed structures due to axial forces and shears are really small compared to distortion due to flexure.

1.3 SIGN CONVENTIONS

In this textbook, the following sign conventions are used:

Moments Clockwise end moments are positive and anticlockwise end moments are negative. In Figure 1.3, $M_{BA} = 20 \text{ kNm}$ and $M_{BC} = -20 \text{ kNm}$

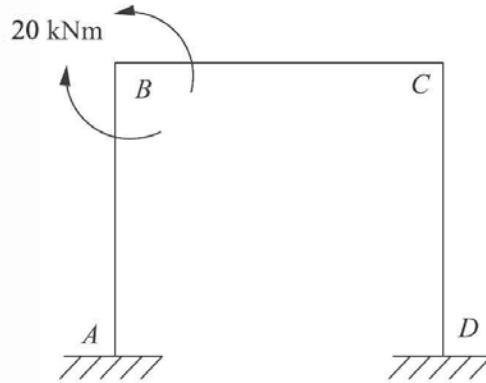


Figure 1.3: Sign convention for moments.

Rotations Clockwise rotation of a tangent to elastic curve is positive rotation and anticlockwise rotation is a negative rotation. In Figure 1.4, θ_1 and θ_4 are positive rotations and θ_2 and θ_3 are negative rotation.

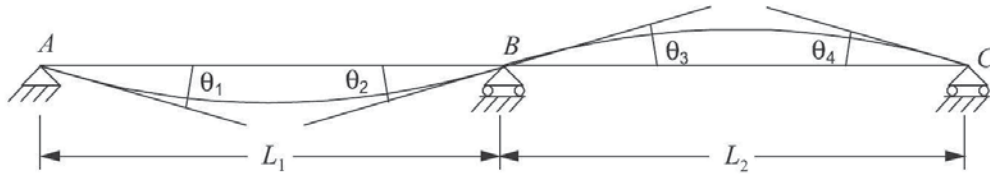


Figure 1.4: Sign convention for rotation.

Settlement Settlement ' Δ ' is positive if right side support is below left side support. ' Δ ' is negative if left side support is below the right side support. Thus, in Figure 1.5, for beam AB, Δ is positive; for beam BC, Δ is negative.

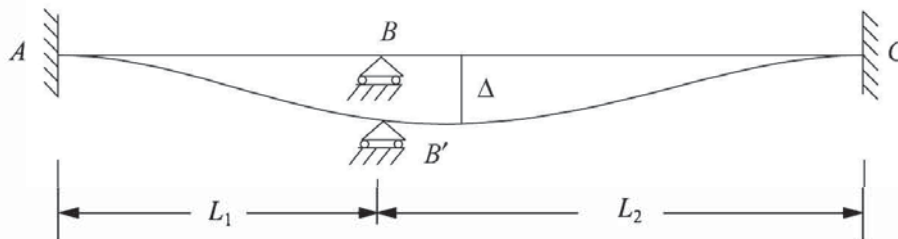


Figure 1.5: Sign convention for settlement.

1.4 DERIVATION OF SLOPE DEFLECTION EQUATIONS

Let AB , shown in Figure 1.6(a), be a member of a rigid structure. After loading it undergoes deformations. Figure 1.6(b) shows deformed shape with all displacements (θ_A , θ_B and Δ) taken in their positive senses. Final moments at end A and end B are M_{AB} and M_{BA} . Now our aim is to derive the relationship between these final end moments and their displacements θ_A , θ_B and Δ .

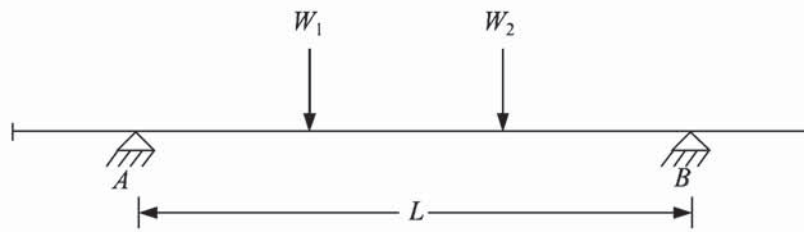


Figure 1.6(a): Original shape of beam.

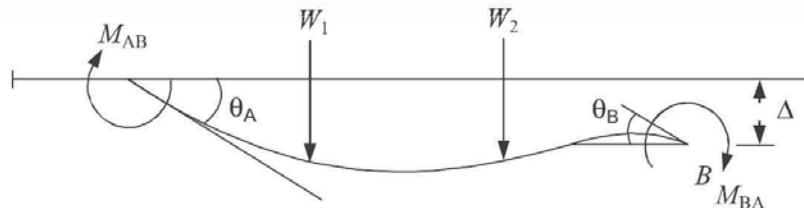


Figure 1.6(b): Deformed shape of the beam.

The development of final moments and deformations are visualised as undergoing in the following stages:

1. Due to given loadings end moments M_{FAB} and M_{FBA} develop without any rotations at ends. These moments are similar to the end moments in a fixed beam and hence are called as *fixed end moments*. In Figure 1.6(c) these are shown in their positive senses. Actually most of the time M_{FAB} will be having a negative value.
2. Settlement Δ takes place without any rotations at ends. This is similar to the settlement of supports in fixed beams. From analysis of fixed beams we know, the end moments developed are $\frac{6EI\Delta}{L^2}$ as shown in Figure 1.6(d).

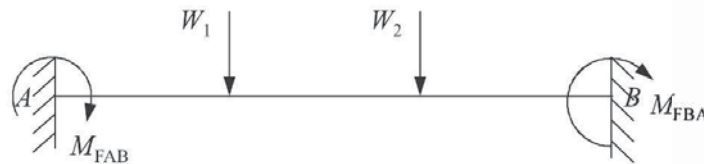


Figure 1.6(c): Fixed end moments developed.

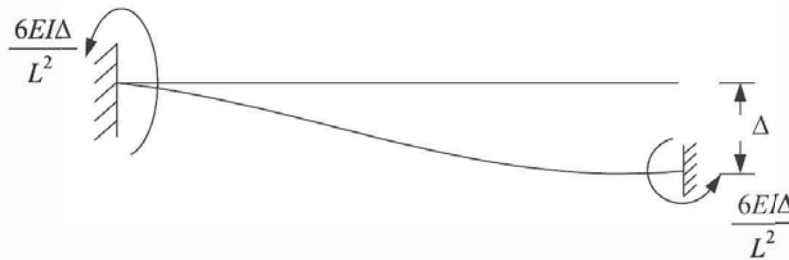


Figure 1.6(d): End moments due to settlement.

3. Moment M'_{AB} comes into play in simply supported beam as shown in Figure 1.6(e) to cause end rotations θ_{A1} and θ_{B1} at A and B respectively.
4. Moments M'_{BA} comes into play in simply supported beam AB, as shown in Figure 1.6(f). The end rotations developed are θ_{A2} and θ_{B2} .

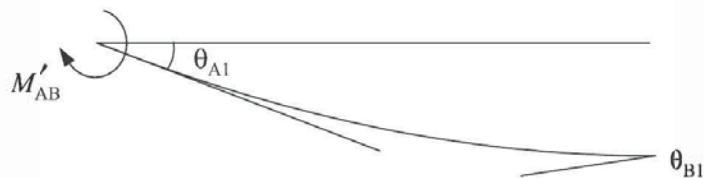


Figure 1.6(e): End rotations due to M'_{AB} .

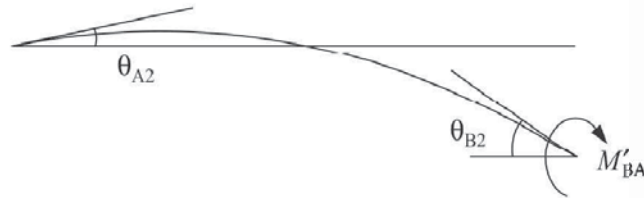


Figure 1.6(f): End rotations due to M'_{BA} .

Moment M'_{AB} and M'_{BA} give final rotations θ_A and θ_B to the beam AB . To find the rotations due to applied moment M in a beam without end rotation (Figure 1.7(a)), conjugate beam method may be used (Figure 1.7(b)).

It may be seen that rotation at loaded end is $\frac{ML}{3EI}$ and at unloaded end is $\frac{ML}{6EI}$.



Figure 1.7(a): Simply supported beam subjected to end moments.

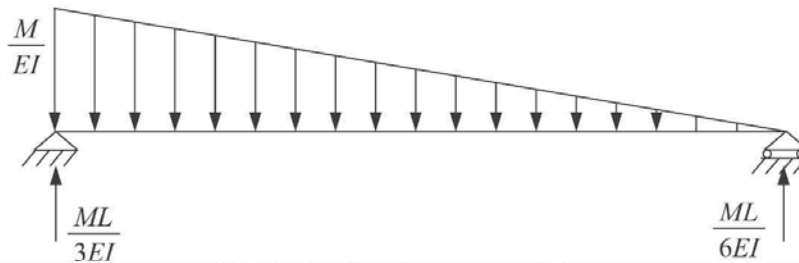


Figure 1.7(b): Conjugate beam.

Hence, referring to Figures 1.6(e) and 1.6(f),

$$\theta_{A1} = \frac{M'_{AB}L}{3EI} \quad \text{and} \quad \theta_{B1} = \frac{M'_{AB}L}{6EI}$$

and

$$\theta_{A2} = \frac{M'_{BA}L}{6EI} \quad \text{and} \quad \theta_{B2} = \frac{M'_{BA}L}{3EI}$$

From Figures 1.6(b), 1.6(e) and 1.6(f), it can be seen that,

$$\theta_A = \theta_{A1} - \theta_{A2} = \left(\frac{M'_{AB}L}{3EI} \right) - \left(\frac{M'_{BA}L}{6EI} \right),$$

and

$$\theta_B = \theta_{B2} - \theta_{B1} = \left(\frac{M'_{BA}L}{3EI} \right) - \left(\frac{M'_{AB}L}{6EI} \right)$$

$$\therefore 2\theta_A + \theta_B = \left(\frac{2M'_{AB}L}{3EI} \right) - \left(\frac{M'_{AB}L}{6EI} \right) = \frac{M'_{AB}L}{6EI} (4 - 1) = \frac{M'_{AB}L}{2EI}$$

$$\therefore M'_{AB} = \frac{2EI}{L} (2\theta_A + \theta_B)$$

Similarly,

$$M'_{BA} = \frac{2EI}{L} (\theta_A + 2\theta_B)$$

Final moments shown in Figure 1.6(b) are the sum of the moments shown in the four stages shown in Figures 1.6(c) to 1.6(f). Referring to Figure 1.6, we write,

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$$\begin{aligned}M_{AB} &= M_{FAB} - \frac{6EI \Delta}{L^2} + M'_{AB} \\&= M_{FAB} - \frac{6EI \Delta}{L^2} + \frac{2EI}{L} (2\theta_A + \theta_B) \\&= M_{FAB} + \frac{2EI}{L} \left(2\theta_A + \theta_B - \frac{3\Delta}{L} \right) \quad \dots(1.1(a))\end{aligned}$$

Similarly,

$$\begin{aligned}M_{BA} &= M_{FBA} - \frac{6EI \Delta}{L^2} + M'_{BA} \\&= M_{FBA} - \frac{6EI \Delta}{L^2} + \frac{2EI}{L} (\theta_A + 2\theta_B) \\&= M_{FBA} + \frac{2EI}{L} \left(\theta_A + 2\theta_B - \frac{3\Delta}{L} \right) \quad \dots(1.1(b))\end{aligned}$$

Equations 1.1(a) and 1.1(b) are known as *Slope deflection equations*.

1.5 APPLICATIONS OF SLOPE DEFLECTION EQUATIONS

Using slope deflection equations, rigid jointed structures can be analysed. The method is illustrated by applying it to the following types of structures:

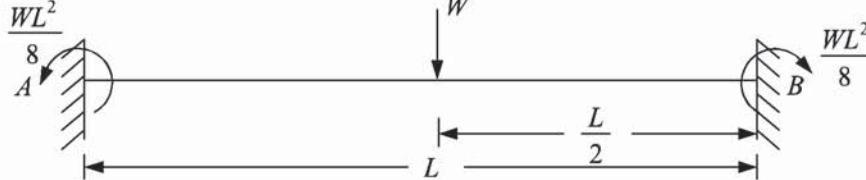
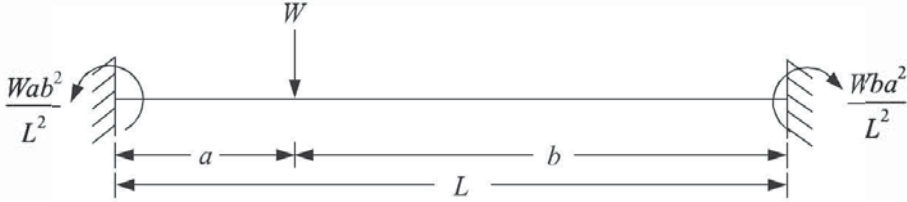
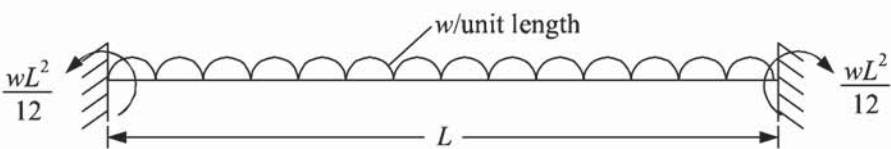
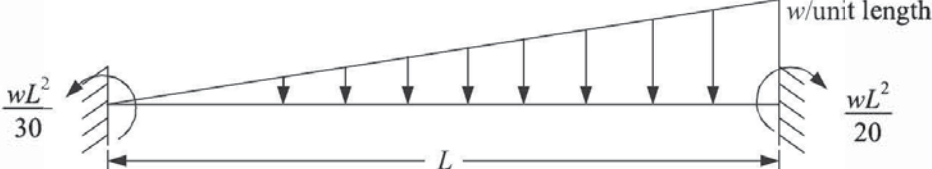
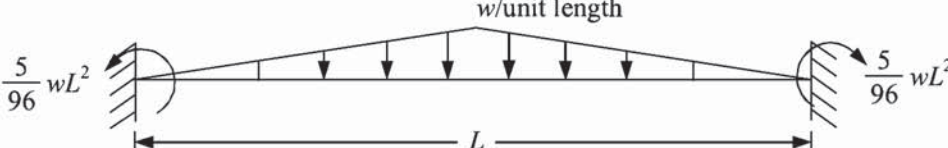
1. Continuous beams
2. Frames without side sway
3. Frames with side sway

1.5.1 Continuous Beams

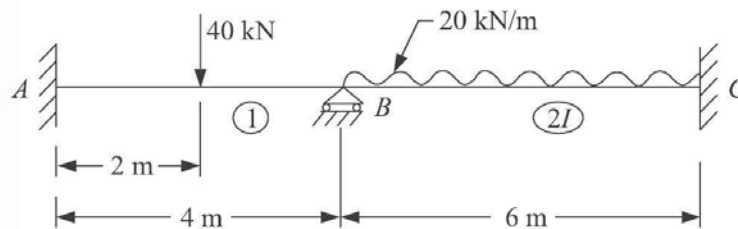
The following steps are involved in the analysis of continuous beams by slope deflection method.

1. Each span of the continuous beam is taken as fixed beam and fixed end moments are noted.
The standard expressions like $\frac{Wab^2}{L^2}$, $\frac{Wba^2}{L^2}$ and $\frac{WL^2}{12}$ for finding end moments in fixed beams may be used (Refer Table 1.1). Clockwise end moments are to be noted as positive moments and anticlockwise as negative moments.
2. Using slope deflection equations write all the end moments. In these equations, some of the rotations and deflections will be unknowns.
3. Write the joint equilibrium equations.
4. Solve the joint equilibrium equations to get the unknown rotations and deflections.
5. Substituting the values of unknowns in slope deflection equations and determine the end moments.
6. Treating each member of the continuous beam as simply supported beam subjected to a given loading and end moments determine the end reactions and draw shear force and bending moment diagrams.

Table 1.1: Expressions for finding fixed end moments

Example 1.1 Analyse the two span continuous beam shown in Figure 1.8 by slope deflection method and draw bending moment, shear force diagrams and elastic curve. (Young's modulus is the same throughout).

**Figure 1.8(a):** Two span continuous beam.

Solution Fixed End Moments

$$M_{FAB} = -\frac{40 \times 4}{8} = -20 \text{ kNm} \quad ; \quad M_{FBA} = 20 \text{ kNm}$$

$$M_{FBC} = -\frac{20 \times 6^2}{12} = -60 \text{ kNm} \quad ; \quad M_{FCB} = 60 \text{ kNm}$$

Slope Deflection Equations

$$M_{AB} = \frac{2EI}{4}(2\theta_A + \theta_B - 0) - 20$$

$$= 0.5 EI\theta_B - 20 \quad \text{[Since, } \theta_A = 0 \text{ as it is fixed end]} \quad \dots(1.2)$$

$$M_{BA} = \frac{2EI}{4}(\theta_A + 2\theta_B - 0) + 20$$

$$= EI\theta_B + 20 \quad \text{[Since, } \theta_A = 0] \quad \dots(1.3)$$

$$M_{BC} = \frac{2E(2I)}{6}(2\theta_B + \theta_C - 0) - 60$$

$$= \frac{4}{3}EI\theta_B - 60 \quad \text{[Since, } \theta_C = 0] \quad \dots(1.4)$$

$$M_{CB} = \frac{2E(2I)}{6}(\theta_B + 2\theta_C - 0) + 60 = \frac{2}{3}EI\theta_B + 60 \quad \text{[Since, } \theta_C = 0] \quad \dots(1.5)$$

Joint Equilibrium Equations

Considering joint B

$$M_{BA} + M_{BC} = 0$$

Substituting the values for M_{BA} and M_{BC} , we get,

$$EI \theta_B + 20 + \frac{4}{3}EI\theta_B - 60 = 0$$

or
$$EI \theta_B = \frac{40}{2.333} = 17.143$$

Substituting this in slope deflection equations, we get,

$$M_{AB} = 0.5 EI \theta_B - 20 = 0.5 \times 17.143 - 20 = -11.428 \text{ kNm}$$

$$M_{BA} = EI \theta_B + 20 = 37.143 \text{ kNm}$$

$$M_{BC} = \frac{4}{3}EI\theta_B - 60 = -37.143 \text{ kNm}$$

$$M_{CB} = \frac{2}{3}EI\theta_B + 60 = 71.429 \text{ kNm}$$

Treating each beam separately with load on it and end moments (Figure 1.8(b))

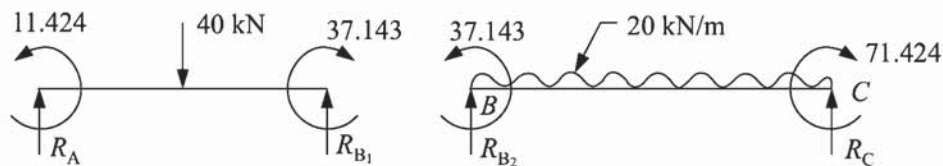


Figure 1.8(b): Free body diagrams.

$$R_A = \frac{11.428 + 40 \times 2 - 37.143}{4} = 13.571 \text{ kN}$$

$$R_{B1} = 40 - 13.571 = 26.429 \text{ kN}$$

$$R_{B2} = \frac{20 \times 6 \times 3 + 37.143 - 71.429}{6} = 54.286 \text{ kN}$$

$$R_C = 20 \times 6 - 54.286 = 65.714 \text{ kN}$$

Hence, shear force diagram is as shown in Figure 1.8(c). Bending moment diagram is drawn keeping in view that tension at bottom is positive and tension at top is negative (Figure 1.8(d)).

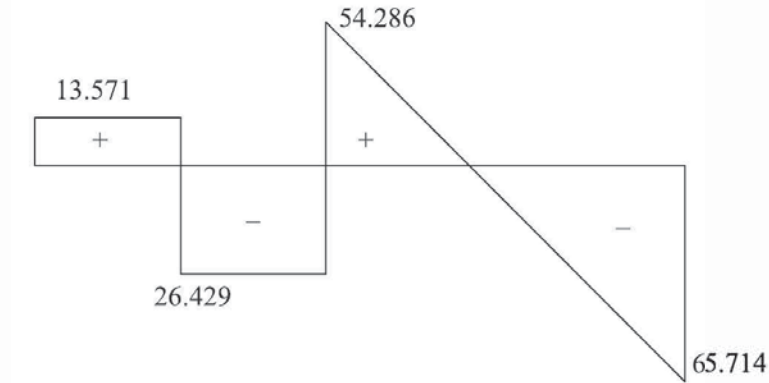


Figure 1.8(c): Shear force diagram (SFD).

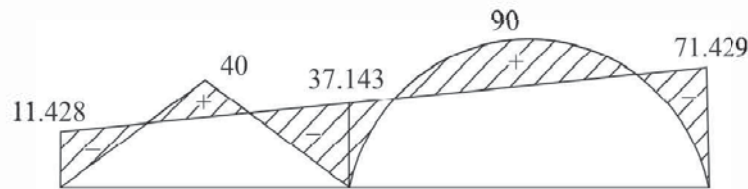


Figure 1.8(d): Bending moment diagram (BMD).

Example 1.2 Analyse the continuous beam shown in Figure 1.9 and draw bending moment diagram.

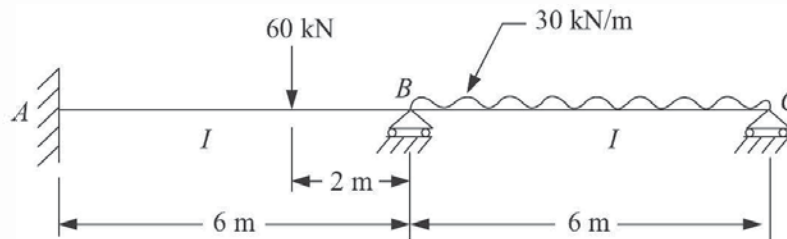


Figure 1.9(a): Continuous beam.

Solution Fixed End Moments

$$M_{FAB} = - \frac{60 \times 4 \times 2^2}{6^2} = -26.67 \text{ kNm}$$

$$M_{FBA} = \frac{60 \times 2 \times 4^2}{6^2} = 53.33 \text{ kNm}$$

$$M_{FBC} = - \frac{30 \times 6^2}{12} = -90 \text{ kNm}$$

$$M_{FCB} = 90 \text{ kNm}$$

Slope Deflection Equations

$$\begin{aligned}
 M_{AB} &= -26.67 + \frac{2EI}{6} (2\theta_A + \theta_B - 0) \\
 &= -26.67 + \frac{1}{3} EI\theta_B \quad (\text{Since, } \theta_A = 0)
 \end{aligned}$$

$$\begin{aligned}
 M_{BA} &= 53.33 + \frac{2EI}{6} (\theta_A + 2\theta_B - 0) \\
 &= 53.33 + \frac{2}{3} EI\theta_B \quad (\text{Since, } \theta_A = 0)
 \end{aligned}$$

$$\begin{aligned}
 M_{BC} &= -90 + \frac{2EI}{6} (2\theta_B + \theta_C - 0) \\
 &= -90 + \frac{2}{3} EI\theta_B + \frac{1}{3} EI\theta_C
 \end{aligned}$$

$$\begin{aligned}
 M_{CB} &= 90 + \frac{2EI}{6} (\theta_B + 2\theta_C - 0) \\
 &= 90 + \frac{1}{3} EI\theta_B + \frac{2}{3} EI\theta_C
 \end{aligned}$$

Equilibrium Equations

$$\Sigma M_B = 0$$

$$M_{BA} + M_{BC} = 0$$

$$i.e., \quad 53.33 + \frac{2}{3} EI\theta_B - 90 + \frac{2}{3} EI\theta_B + \frac{1}{3} EI\theta_C = 0$$

$$4 EI\theta_B + EI\theta_C = 110 \quad \dots(1.6)$$

$$\Sigma M_C = 0, \text{ gives } M_{CB} = 0$$

$$i.e., \quad 90 + \frac{1}{3} EI\theta_B + \frac{2}{3} EI\theta_C = 0$$

$$\text{or} \quad EI\theta_B + 2 EI\theta_C = -270 \quad \dots(1.7)$$

Subtracting Eqn. (1.7) from twice Eqn. (1.6), we get,

$$7 EI\theta_B = 490$$

$$EI\theta_B = 70 \quad \dots(1.8)$$

Therefore, from Eqn. (1.6),

$$EI\theta_C = 110 - 4 \times 70 = -170 \quad \dots(1.9)$$

End Moments

Substituting the values of $EI\theta_B$ and $EI\theta_C$ in slope deflection equations, we get,

$$M_{AB} = -26.67 + \frac{1}{3} (70) = -3.33 \text{ kNm}$$

$$M_{BA} = 53.33 + \frac{2}{3} (70) = 100 \text{ kNm}$$

$$M_{BC} = -90 + \frac{2}{3}(70) + \frac{1}{3}(-170) = -100 \text{ kNm}$$

$$M_{CB} = 90 + \frac{1}{3}(70) + \frac{2}{3}(-170) = 0$$

The sign conventions for these moments is clockwise direction positive and anticlockwise direction negative. Hence, the moments are as shown in Figure 1.9(b). But while drawing bending moment diagrams sign convention followed is tension at bottom positive and tension at top negative. Free moments (moment in a simply supported beam for given loading) give tension at bottom and hence are positive. End moments give tension at top and hence they are negative moments. Since, we need difference of these diagrams, to get the final diagram, we draw them on the same side and mark the difference diagram as the final diagram.

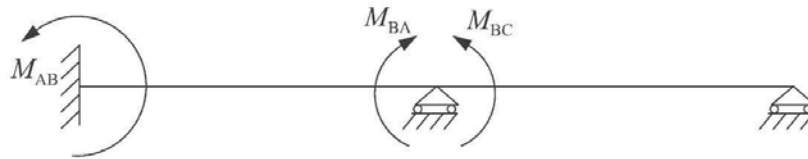


Figure 1.9(b): Final moments.

Free moment diagram in AB is a triangle with maximum ordinate under load, its magnitude being $\frac{60 \times 4 \times 2}{6} = 80 \text{ kNm}$

Free moment diagram in BC is a symmetric parabola with maximum ordinate,

$$= \frac{30 \times 6^2}{8} = 135 \text{ kNm}$$

Hence, BMD is as shown in Figure 1.9(c).

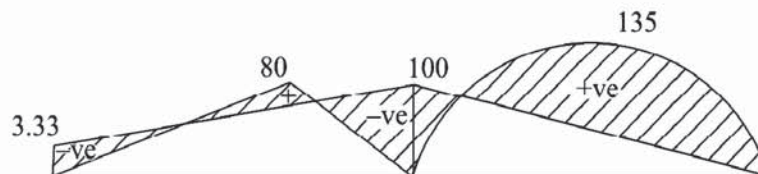


Figure 1.9(c): Bending moment diagram.

Example 1.3 Analyse the continuous beam shown in Figure 1.10 by slope deflection method and draw bending moment diagram.

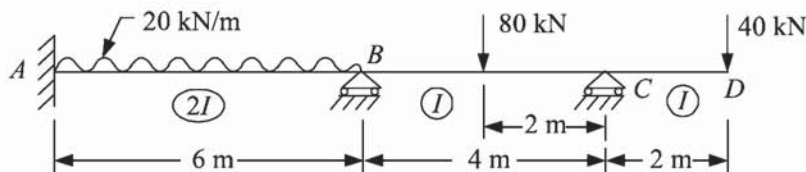


Figure 1.10: Continuous beam.

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Solution Fixed End Moments

$$M_{FAB} = -\frac{20 \times 6^2}{12} = -60 \text{ kNm}$$

$$M_{FBA} = 60 \text{ kNm}$$

$$M_{FBC} = -\frac{80 \times 4}{8} = -40 \text{ kNm}$$

$$M_{FCB} = 40 \text{ kNm}$$

Slope Deflection Equations

$$\begin{aligned} M_{AB} &= -60 + \frac{2E(2I)}{6}(2\theta_A + \theta_B - 0) \\ &= -60 + \frac{2}{3}EI\theta_B \quad (\text{Since, } \theta_A = 0) \end{aligned}$$

$$M_{BA} = 60 + \frac{2E(2I)}{6}(\theta_A + 2\theta_B - 0) = 60 + \frac{4}{3}EI\theta_B$$

$$\begin{aligned} M_{BC} &= -40 + \frac{2EI}{4}(2\theta_B + \theta_C - 0) \\ &= -40 + EI\theta_B + 0.5 EI\theta_C \end{aligned}$$

$$\begin{aligned} M_{CB} &= 40 + \frac{2EI}{4}(\theta_B + 2\theta_C - 0) \\ &= 40 + 0.5 EI\theta_B + EI\theta_C \end{aligned}$$

Equilibrium Equations

$$\Sigma M_B = 0$$

$$M_{BA} + M_{BC} = 0$$

$$60 + \frac{4}{3}EI\theta_B - 40 + EI\theta_B + 0.5 EI\theta_C = 0$$

$$2.333 EI\theta_B + 0.5 EI\theta_C = -20 \quad \dots(1.10)$$

$$\Sigma M_C = 0$$

$$M_{CB} + M_{CD} = 0$$

But, $M_{CD} = -40 \times 2 = -80$

$\therefore M_{CB} = 80 \text{ kNm}$

i.e., $40 + 0.5 EI\theta_B + EI\theta_C = 80$

or $0.5 EI\theta_B + EI\theta_C = 40 \quad \dots(1.11)$

Subtracting Eqn. (1.11) from twice Eqn. (1.10), we get,

$$(4.667 - 0.5) EI\theta_B = -80$$

or $EI\theta_B = 19.2 \quad \dots(1.12)$

From Eqn. (1.11), we get,

$$0.5(-19.2) + EI \theta_C = 40$$

or $EI \theta_C = 49.6$... (1.13)

Final Moments

Substituting the values of $EI \theta_B$ and $EI \theta_C$, in slope deflection equations, we get,

$$M_{AB} = -60 + \frac{2}{3}(-19.2) = -72.8 \text{ kNm}$$

$$M_{BA} = 60 + \frac{4}{3}(-19.2) = 34.4 \text{ kNm}$$

$$M_{BC} = -40 + (-19.2) + 0.5 \times 49.6 = -34.4 \text{ kNm}$$

$$M_{CB} = 80 \text{ kNm}$$

Example 1.4 Analyse the continuous beam shown in Figure 1.11 by slope deflection method, if joint B sinks by 10 mm. Given $EI = 4000 \text{ kNm}^2$. Draw bending moment and shear force diagrams. Draw elastic curve also.

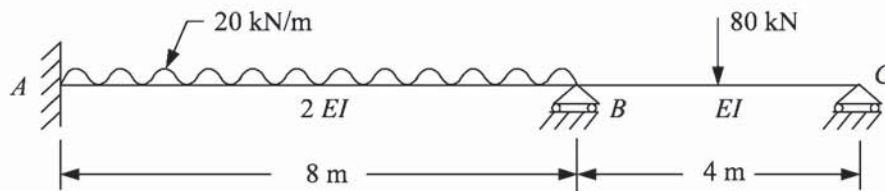


Figure 1.11(a): Continuous beam.

Solution Fixed End Moments

$$M_{FAB} = -\frac{20 \times 8^2}{12} = -106.67 \text{ kNm}$$

$$M_{FBA} = \frac{20 \times 8^2}{12} = 106.67 \text{ kNm}$$

$$M_{FBC} = -\frac{80 \times 4}{8} = -40 \text{ kNm}$$

$$M_{FCB} = \frac{80 \times 4}{8} = 40 \text{ kNm}$$

Slope Deflection Equations

$$\begin{aligned} M_{AB} &= M_{FAB} + \frac{2(2EI)}{8} \left(2\theta_A + \theta_B - \frac{3\Delta}{L} \right) \\ &= -106.67 + 0.5EI\theta_B - \frac{EI}{2} \frac{3(0.010)}{8}, \text{ (Since, } \Delta = 10 \text{ mm} = 0.01 \text{ m and } \theta_A = 0) \\ &= -106.67 + 0.5EI\theta_B - \left(\frac{4000}{2} \right) \left(\frac{0.010}{8} \right), \text{ (Since, } EI = 4000 \text{ kNm}^2) \\ &= 0.5EI\theta_B - 114.17 \end{aligned}$$

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$$\begin{aligned}
 M_{BA} &= 106.67 + \frac{2(2EI)}{8} \left(2\theta_A + \theta_B - \frac{3\Delta}{L} \right) \\
 &= 106.67 + EI \theta_B - 0.5 EI \left(\frac{3\Delta}{8} \right) \\
 &= 106.67 + EI \theta_B - 0.5 \times 4000 \times \frac{3 \times 0.010}{8} \\
 &= EI \theta_B + 99.17
 \end{aligned}$$

$$\begin{aligned}
 M_{BC} &= -30 + \frac{2EI}{4} \left(2\theta_B + \theta_C - \frac{3\Delta}{L} \right) \\
 &= -30 + EI \theta_B + 0.5 EI \theta_C - \frac{2}{4} \times 4000 \times 3 \frac{(-0.010)}{4}, \\
 &\hspace{15em} (\text{Since, } \Delta = -10 \text{ mm} = -0.010 \text{ m}) \\
 &= EI \theta_B + 0.5 EI \theta_C - 15
 \end{aligned}$$

$$\begin{aligned}
 M_{CB} &= 30 + \frac{2EI}{4} \left(\theta_B + 2\theta_C - \frac{3\Delta}{L} \right) \\
 &= 30 + 0.5 EI \theta_B + EI \theta_C - \frac{2}{4} \times 4000 \times 3 \frac{(-0.010)}{4} \\
 &= 0.5 EI \theta_B + EI \theta_C + 45
 \end{aligned}$$

Equilibrium Equations

$$\Sigma M_B = 0$$

$$M_{BA} + M_{BC} = 0$$

$$EI \theta_B + 99.17 + EI \theta_B + 0.5 EI \theta_C - 15 = 0$$

$$\text{or } 2 EI \theta_B + 0.5 EI \theta_C = -84.17 \quad \dots(1.14)$$

$$\Sigma M_C = 0$$

$$M_{CB} = 0$$

$$\text{i.e., } 0.5 EI \theta_B + EI \theta_C + 45 = 0$$

$$\text{or } 0.5 EI \theta_B + EI \theta_C = -45 \quad \dots(1.15)$$

$$\text{From Eqn. (1.14), } 4 EI \theta_B + EI \theta_C = -168.34 \quad \dots(1.16)$$

Subtracting Eqn. (1.15) from Eqn. (1.16), we get,

$$3.5 EI \theta_B = -123.34$$

$$EI \theta_B = -35.24$$

Substituting it in Eqn. (1.15), we get,

$$0.5 (-35.24) + EI \theta_C = -45$$

$$\text{or } EI \theta_C = -27.38$$

Substituting the values of $EI \theta_B$ and $EI \theta_C$ in slope deflection equations, we get,

$$M_{AB} = 0.5(-35.24) - 114.17 = -131.79 \text{ kNm}$$

$$M_{BA} = -35.24 + 99.17 = 63.93 \text{ kNm}$$

$$M_{BC} = 0.5(-35.24) + (-27.38) - 15 = -60 \text{ kNm}$$

$$M_{CB} = 0.5(-35.24) + (-27.38) + 45 = 0$$

Bending Moment Diagram (BMD)

Free moment diagram in span AB is a symmetric parabola with maximum ordinate

$$= \frac{20 \times 8^2}{8} = 160 \text{ kNm}$$

Free moment diagram in span BC is a triangle with maximum ordinate at centre of span of magnitude

$$= \frac{60 \times 4}{4} = 60 \text{ kNm}$$

The end moments create tension at the top. Hence, bending moment diagram is as shown in Figure 1.11(b).

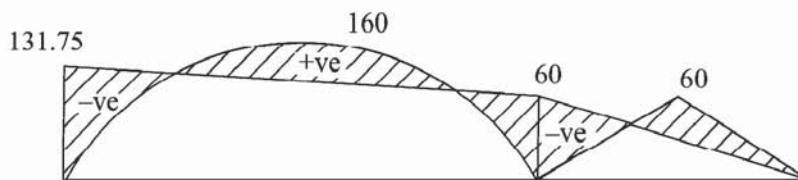


Figure 1.11(b): Bending moment diagram.

Shear Force Diagram (SFD)

For calculating support reactions each span is treated separately as loaded with given loading and end moments as shown in Figure 1.11(c).

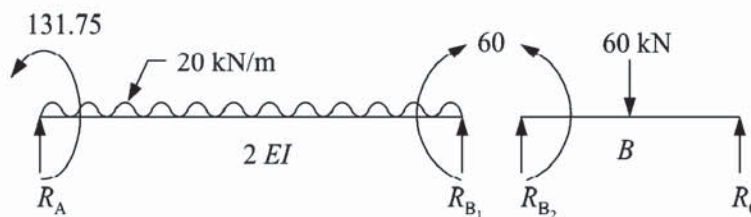


Figure 1.11(c): Free body diagram.

$$R_A = \frac{20 \times 8 \times 4 + 131.75 - 60}{8} = 88.961 \text{ kN}$$

$$R_{B1} = 20 \times 8 - 88.961 = 71.039 \text{ kN}$$

$$R_{B2} = \frac{60 \times 2 + 60}{4} = 45 \text{ kN}$$

Hence, $R_B = R_{B1} + R_{B2} = 71.039 + 45 = 116.039 \text{ kN}$

$$R_C = 60 - 45 = 15 \text{ kN}$$

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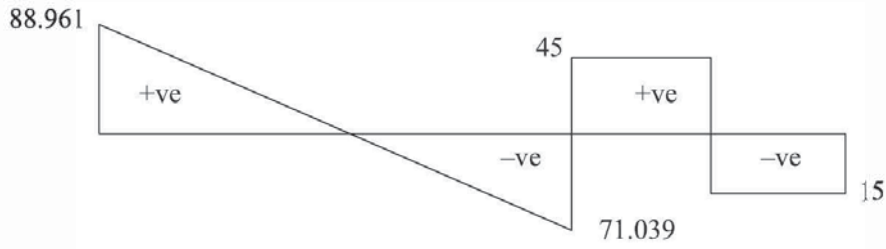


Figure 1.11(d): Shear force diagram.



Figure 1.11(e): Elastic curve.

Hence, SFD is as shown in Figure 1.11(d).

Elastic curve is shown in Figure 1.11(e).

Example 1.5 Analyse the continuous beam *ABCD* shown in Figure 1.12(a) by slope deflection method and draw bending moment diagram.

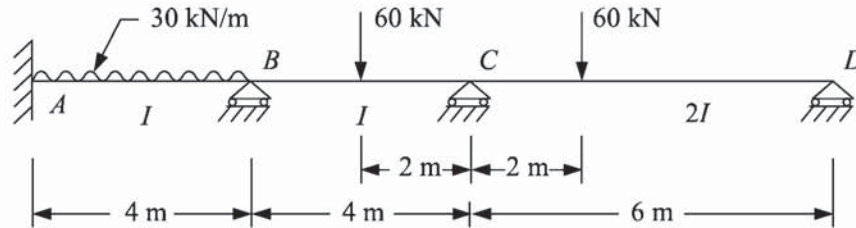


Figure 1.12(a): Continuous beam.

Solution Fixed End Moments

$$M_{FAB} = -\frac{30 \times 4^2}{12} = -40 \text{ kNm}$$

$$M_{FBA} = 40 \text{ kNm}$$

$$M_{FBC} = -\frac{60 \times 4}{8} = -30 \text{ kNm}$$

$$M_{FCB} = 30 \text{ kNm}$$

$$M_{FCD} = -\frac{60 \times 2 \times 4^2}{6^2} = -53.333 \text{ kNm}$$

$$M_{FDC} = \frac{60 \times 2^2 \times 4}{6^2} = 26.667 \text{ kNm}$$

Slope Deflection Equations

$$M_{AB} = -40 + \frac{2EI}{4} (2\theta_A + \theta_B - 0)$$

$$= -40 + 0.5 EI \theta_B$$

(Since, $\theta_A = 0$)

$$\begin{aligned}
 M_{BA} &= 40 + \frac{2EI}{4}(\theta_A + 2\theta_B - 0) \\
 &= 40 + EI \theta_B \quad (\text{Since, } \theta_A = 0)
 \end{aligned}$$

$$\begin{aligned}
 M_{BC} &= -30 + \frac{2EI}{4}(\theta_C + 2\theta_B - 0) \\
 &= -30 + EI \theta_B + 0.5 EI \theta_C
 \end{aligned}$$

$$M_{CB} = 30 + \frac{2EI}{4}(\theta_B + 2\theta_C - 0) = 30 + 0.5 EI \theta_B + EI \theta_C$$

$$\begin{aligned}
 M_{CD} &= -53.333 + \frac{2EI}{6}(2\theta_C + \theta_D - 0) \\
 &= -53.333 + 0.667 EI \theta_C + 0.333 EI \theta_D
 \end{aligned}$$

$$\begin{aligned}
 M_{DC} &= 26.667 + \frac{2EI}{6}(\theta_C + 2\theta_D - 0) \\
 &= 26.667 + 0.333 EI \theta_C + 0.667 EI \theta_D
 \end{aligned}$$

Thus, it is a system of three degrees of freedom, *i.e.*, a system with three unknown displacement components at joints. We need three equilibrium equations to determine them.

Equilibrium Equations

$$\Sigma M_B = 0$$

$$M_{BA} + M_{BC} = 0$$

$$40 + EI \theta_B - 30 + EI \theta_B + 0.5 EI \theta_C = 0$$

$$i.e., \quad 2 EI \theta_B + 0.5 EI \theta_C = -10 \quad \dots(1.17)$$

$$\Sigma M_C = 0$$

$$M_{CB} + M_{CD} = 0$$

$$30 + 0.5 EI \theta_B + EI \theta_C - 53.333 + 0.667 EI \theta_C + 0.333 EI \theta_D = 0$$

Multiply throughout by 3 and rearranging the terms, we get,

$$1.5 EI \theta_B + 5 EI \theta_C + EI \theta_D = 70 \quad \dots(1.18)$$

$$\Sigma M_D = 0$$

$$M_{DC} = 0$$

$$i.e., \quad 26.667 + 0.333 EI \theta_C + 0.667 EI \theta_D = 0$$

$$or \quad EI \theta_C + 2EI \theta_D = -80 \quad \dots(1.19)$$

Multiplying Eqn. (1.18) by 2, we get,

$$3 EI \theta_B + 10 EI \theta_C + 2 EI \theta_D = 140 \quad \dots(1.20)$$

Subtracting Eqn. (1.19) from Eqn. (1.20), we get,

$$3 EI \theta_B + 9 EI \theta_C = 220 \quad \dots(1.21)$$

Multiplying Eqn. (1.17) by 1.5, we get,

$$3 EI \theta_B + 0.75 EI \theta_C = -15 \quad \dots(1.22)$$

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Subtracting Eqn. (1.22) from Eqn. (1.21), we get,

$$8.25 EI \theta_C = 235$$

$$EI \theta_C = 28.485$$

From Eqn. (1.21), we get,

$$3 EI \theta_B + 9(28.485) = 220$$

or

$$EI \theta_B = -12.121$$

From Eqn. (1.19), we get,

$$28.485 + 2 EI \theta_D = -80$$

or

$$EI \theta_D = -54.243.$$

Note. Calculator may be used to solve three simultaneous equations.

Final Moments

Substituting the values of $EI \theta_B$, $EI \theta_C$ and $EI \theta_D$ in slope deflection equations, final moments may be obtained.

$$M_{AB} = -40 + 0.5(-12.121) = -46.061 \text{ kNm}$$

$$M_{BA} = 40 - 12.121 = 27.879 \text{ kNm}$$

$$M_{BC} = -27.879 \text{ kNm}$$

$$M_{CB} = 30 + 0.5(-12.121) + 28.485 = 52.425 \text{ kNm}$$

$$M_{CD} = -54.425 \text{ kNm}$$

$$M_{DC} = 0$$

Bending Moment Diagram

Free moment diagram for beam AB is a symmetric parabola with maximum ordinate

$$= \frac{30 \times 4^2}{8} = 60 \text{ kNm}$$

Free moment diagram for beam BC is a symmetric triangle with maximum ordinate

$$= \frac{60 \times 4}{4} = 60 \text{ kNm}$$

Free moment diagram for beam CD is a triangle with maximum ordinate under the load and its magnitude is

$$= \frac{60 \times 2 \times 4}{6} = 80 \text{ kNm}$$

End moments create tension at top. Hence, bending moment diagram is as shown in Figure 1.12(b).

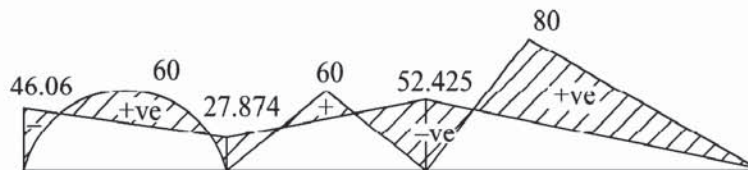


Figure 1.12(b): Bending moment diagram.

Example 1.6 Formulate slope deflection equations and equilibrium equations for the continuous beam shown in Figure 1.13. Moment of inertia is same throughout.

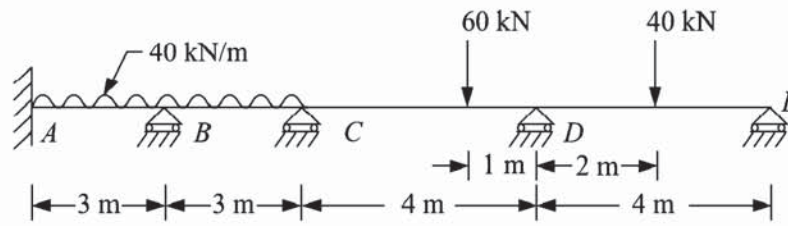


Figure 1.13: Continuous beam.

Solution Fixed End Moments

$$M_{FAB} = -\frac{40 \times 3^2}{12} = -30 \text{ kNm}$$

$$M_{FBA} = 30 \text{ kNm}$$

$$M_{FBC} = -30 \text{ kNm}$$

$$M_{FCB} = 30 \text{ kNm}$$

$$M_{FCD} = -\frac{60 \times 3 \times 1^2}{4^2} = -11.25 \text{ kNm}$$

$$M_{FDC} = \frac{60 \times 3^2 \times 1}{4^2} = 33.75 \text{ kNm}$$

$$M_{FDE} = -\frac{40 \times 4}{8} = -20 \text{ kNm}$$

$$M_{FED} = 20 \text{ kNm}$$

Slope Deflection Equations

$$M_{AB} = -30 + \frac{2EI}{3}(2\theta_A + \theta_B - 0) = -30 + \frac{2EI}{3}(\theta_B) \quad (\text{Since, } \theta_A = 0)$$

$$M_{BA} = 30 + \frac{2EI}{3}(\theta_A + 2\theta_B - 0) = 30 + \frac{4EI}{3}(\theta_B)$$

$$M_{BC} = -30 + \frac{2EI}{3}(2\theta_B + \theta_C - 0) = -30 + \frac{4EI}{3}(\theta_B) + \frac{2EI}{3}(\theta_C)$$

$$M_{CB} = 30 + \frac{2EI}{3}(\theta_B + 2\theta_C - 0) = 30 + \frac{2EI}{3}\theta_B + \frac{4EI}{3}\theta_C$$

$$M_{CD} = -11.25 + \frac{2EI}{4}(2\theta_C + \theta_D - 0) = -11.25 + EI\theta_C + 0.5EI\theta_D$$

$$M_{DC} = 33.75 + \frac{2EI}{4}(\theta_C + 2\theta_D - 0) = 33.75 + 0.5EI\theta_C + EI\theta_D$$

$$M_{DE} = -20 + \frac{2EI}{4}(2\theta_D + \theta_E - 0) = -20 + EI\theta_D + 0.5EI\theta_E$$

$$M_{ED} = 20 + \frac{2EI}{4}(\theta_D + 2\theta_E - 0) = 20 + 0.5EI\theta_D + EI\theta_E$$

Equilibrium Equations

$$\Sigma M_B = 0$$

$$M_{BA} + M_{BC} = 0$$

$$30 + \frac{4EI}{3}\theta_B - 30 + \frac{4EI}{3}\theta_B + \frac{2EI}{3}\theta_C = 0$$

$$\text{i.e., } \frac{8}{3}EI\theta_B + \frac{2}{3}EI\theta_C = 0$$

$$\text{or } 4EI\theta_B + EI\theta_C = 0 \quad \dots(1.23)$$

$$\Sigma M_C = 0$$

$$M_{CB} + M_{CD} = 0$$

$$30 + \left(\frac{2EI}{3}\right)\theta_B + \left(\frac{4EI}{3}\right)\theta_C - 11.25 + EI\theta_C + 0.5EI\theta_D = 0$$

$$\frac{2}{3}EI\theta_B + \frac{7}{3}EI\theta_C + 0.5EI\theta_D = 11.25 - 30$$

$$2EI\theta_B + 7EI\theta_C + 1.5EI\theta_D = -56.25 \quad \dots(1.24)$$

$$\Sigma M_D = 0$$

$$M_{DC} + M_{DE} = 0$$

$$33.75 + 0.5EI\theta_C + EI\theta_D - 20 + EI\theta_D + 0.5EI\theta_E = 0$$

$$0.5EI\theta_C + 2EI\theta_D + 0.5EI\theta_E = -13.75 \quad \dots(1.25)$$

$$\Sigma M_E = 0$$

$$20 + 0.5EI\theta_D + EI\theta_E = 0$$

$$\text{or } 0.5EI\theta_D + EI\theta_E = -20 \quad \dots(1.26)$$

Thus, equilibrium equations are:

$$\begin{bmatrix} 4 & 1 & 0 & 0 \\ 2 & 7 & 1.5 & 0 \\ 0 & 0.5 & 2 & 0.5 \\ 0 & 0 & 0.5 & 1.0 \end{bmatrix} \begin{Bmatrix} EI\theta_B \\ EI\theta_C \\ EI\theta_D \\ EI\theta_E \end{Bmatrix} = \begin{Bmatrix} 0 \\ -56.25 \\ -13.75 \\ -20 \end{Bmatrix}$$

1.5.2 Analysis of Frames Without Sway

The side movement of the end of a column in a frame is called *sway*. Figure 1.14(a) shows frame with sway when loaded. Both columns are having same sway (Δ), since, axial deformation of beam *BC* is assumed negligible. In the frames shown in Figures 1.14(b) and (c), sway is prevented by unyielding supports provided at the beam level. Frames shown in Figures 1.14(d), (e) and (f) also do not have sway, because of geometric as well as load symmetry about the vertical axis. As a first step in the analysis, in this article, frames without sway are considered. The procedure followed in the analysis of such frames is same as that followed for continuous beams.

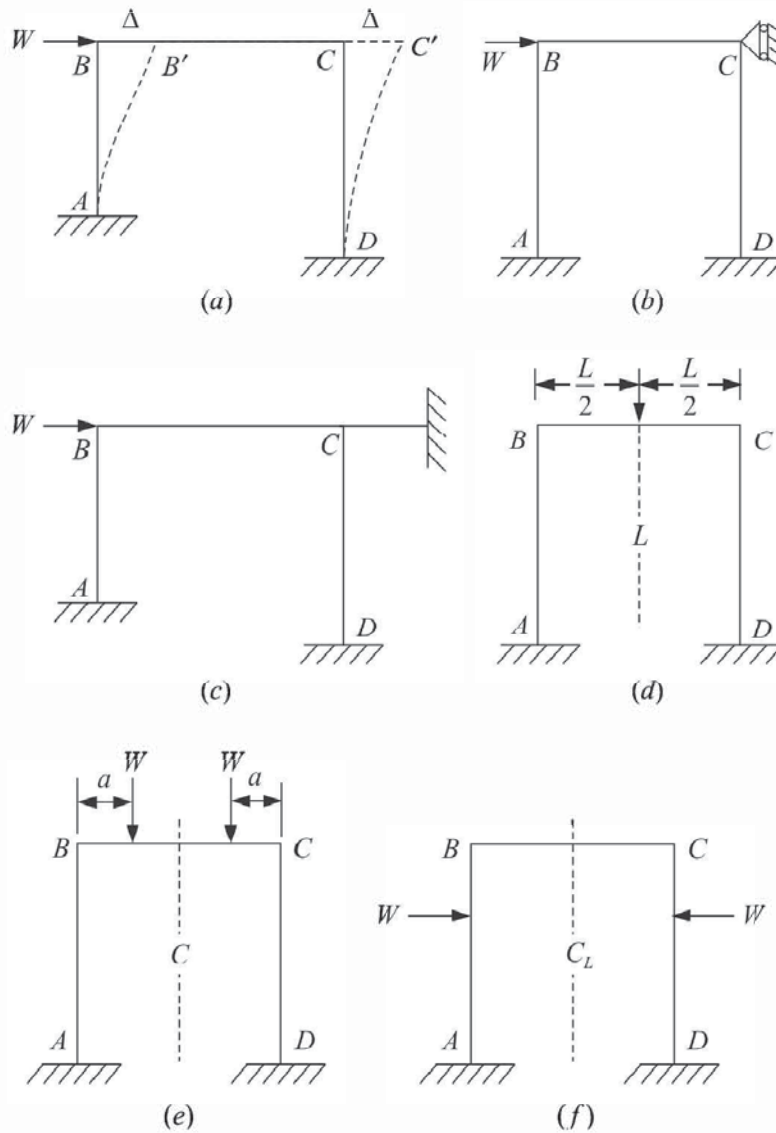


Figure 1.14(a): Frame with sway. **(b)** Sway prevented by unyielding support. **(c)** Sway prevented by unyielding support. **(d)** No sway due to symmetry. **(e)** No sway due to symmetry. **(f)** No sway due to symmetry.

Example 1.7 Analyse the frame shown in Figure 1.15(c) by slope deflection method and draw bending moment diagram.

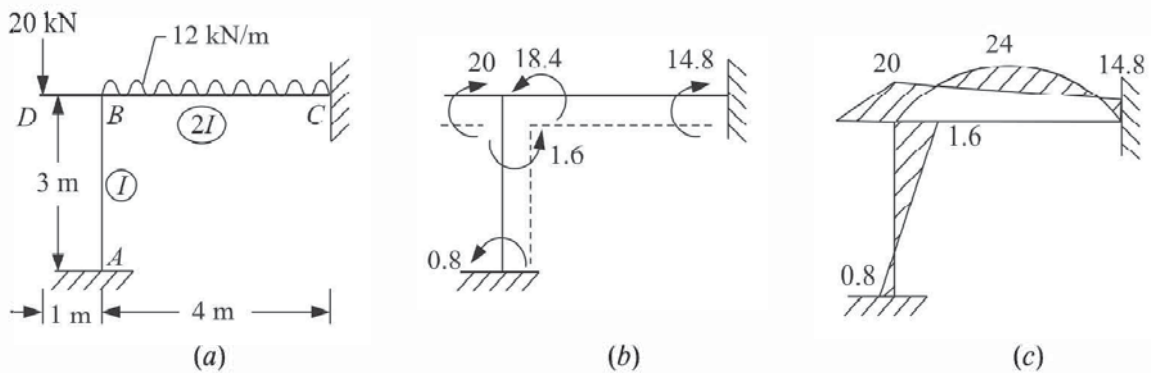


Figure 1.15(a): Given frame. **(b)** End moments. **(c)** Bending moment diagram.

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Solution Fixed End Moments

$$M_{FAB} = M_{FBA} = 0$$

$$M_{FBD} = 20 \times 1 = 20 \text{ kNm} \quad (\text{Since, } BD \text{ is a cantilever})$$

$$M_{FBC} = -\frac{12 \times 4^4}{12} = -16 \text{ kNm}$$

$$M_{FCB} = 16 \text{ kNm}$$

Slope Deflection Equations

$$M_{AB} = 0 + \frac{2EI}{3} (2\theta_A + \theta_B - 0) = \frac{2}{3} EI \theta_B, \quad (\text{Since, } \theta_A = 0)$$

$$M_{BA} = 0 + \frac{2EI}{3} (\theta_A + 2\theta_B - 0) = \frac{4}{3} EI \theta_B$$

$$M_{BC} = -16 + \frac{2E(2I)}{4} (2\theta_B + \theta_C) = -16 + 2EI \theta_B, \quad (\text{Since, } \theta_C = 0)$$

$$M_{CB} = 16 + \frac{2E(2I)}{4} (\theta_B + 2\theta_C) = 16 + EI \theta_B, \quad (\text{Since, } \theta_C = 0)$$

Equilibrium Equations

$$\Sigma M_B = 0, \text{ gives}$$

$$M_{BA} + M_{BC} + M_{BD} = 0$$

$$\frac{4}{3} EI \theta_B - 16 + 2EI \theta_B + 20 = 0$$

$$\frac{10}{3} EI \theta_B = -4$$

$$EI \theta_B = -1.2$$

Final Moments

$$M_{AB} = \frac{2}{3} EI \theta_B = \frac{2}{3} (-1.2) = -0.8 \text{ kNm}$$

$$M_{BA} = \frac{4}{3} (-1.2) = -1.6 \text{ kNm}$$

$$M_{BC} = -16 + 2(-1.2) = -18.4 \text{ kNm}$$

$$M_{CB} = 16 + (-1.2) = 14.8 \text{ kNm}$$

Bending Moment Diagram

Free moment diagram in span BC is a symmetric parabola with maximum value

$$= \frac{12 \times 4^4}{8} = 24 \text{ kNm}$$

Noting that clockwise end moments are positive and that anticlockwise end moments are negative, these values are marked in Figure 1.15(b). Bending moment diagram is drawn, taking tension on dotted side as positive and tension on other side negative [Figure 1.15(b)]. First free moment diagram is drawn on compression side. Then end moment diagram is drawn on tension

side of the member so that difference diagram, which is the final bending moment diagram can be easily marked. It may be carefully noted that, if the end moments are marked in their sense on members around the joint [as shown in Figure 1.15(b)] arrow head comes out on tension side. Figure 1.15(c) shows final bending moment diagram.

Example 1.8 Analyse the frame shown in Figure 1.16(a) by slope deflection method and draw bending moment diagram. Flexural rigidity (EI) is same for all members.

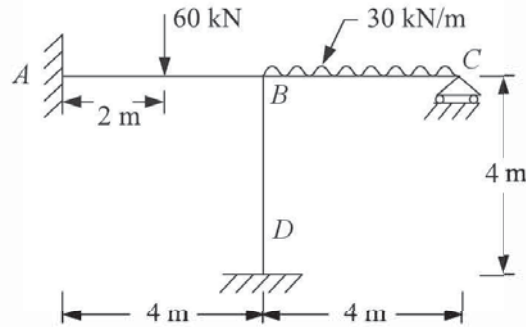


Figure 1.16(a): Given frame.

Solution Fixed End Moments

$$M_{FAB} = -\frac{120 \times 4}{8} = -60 \text{ kNm}$$

$$M_{FBA} = 60 \text{ kNm}$$

$$M_{FBC} = -\frac{30 \times 4^2}{12} = -40 \text{ kNm}$$

$$M_{FCB} = 40 \text{ kNm}$$

$$M_{FBD} = M_{FDB} = 0$$

Slope Deflection Equations

$$M_{AB} = -60 + \frac{2EI}{4}(2\theta_A + \theta_B - 0) = -60 - 0.5 EI \theta_B, \quad (\text{Since, } \theta_A = 0)$$

$$M_{BA} = 60 + \frac{2EI}{4}(\theta_A + 2\theta_B - 0) = 60 + EI \theta_B$$

$$M_{BC} = -40 + \frac{2EI}{4}(2\theta_B + \theta_C) = -40 + EI \theta_B + 0.5 EI \theta_C$$

$$M_{CB} = 40 + \frac{2EI}{4}(\theta_B + 2\theta_C) = 40 + 0.5 EI \theta_B + EI \theta_C$$

$$M_{BD} = 0 + \frac{2EI}{4}(2\theta_B + \theta_D) = EI \theta_B \quad (\text{Since, } \theta_D = 0)$$

$$M_{DB} = 0 + \frac{2EI}{4}(\theta_B + 2\theta_D) = 0.5 EI \theta_B \quad (\text{Since, } \theta_D = 0)$$

Equilibrium Equations

$$\Sigma M_B = 0, \text{ gives}$$

$$M_{BA} + M_{BC} + M_{BD} = 0$$

$$60 + EI \theta_B - 40 + EI \theta_B + 0.5 EI \theta_C + EI \theta_B = 0$$

$$3 EI \theta_B + 0.5 EI \theta_C = -20 \quad \dots(1.27)$$

$$\Sigma M_C = 0, \text{ gives}$$

$$M_{CB} = 0$$

$$40 + 0.5 EI \theta_B + EI \theta_C = 0$$

$$\text{or} \quad 0.5 EI \theta_B + EI \theta_C = -40 \quad \dots(1.28)$$

Subtracting Eqn. (1.28) from twice Eqn. (1.27), we get,

$$5.5 EI \theta_B = 0$$

$$\text{or} \quad EI \theta_B = 0$$

$$\therefore \text{ From Eqn. (1.28),} \quad EI \theta_C = -40$$

Final Moments

Substituting the values of $EI \theta_B$ and $EI \theta_C$ in slope deflection equations, we get,

$$M_{AB} = -60 \text{ kNm}$$

$$M_{BA} = 60 \text{ kNm}$$

$$M_{BC} = -40 + 0.5(-40) = -60 \text{ kNm}$$

$$M_{CB} = 40 + 0 - 40 = 0$$

$$M_{BD} = 0$$

$$M_{DB} = 0$$

Bending Moment Diagram

Free moment diagram in span AB is a symmetric parabola with maximum ordinate

$$= \frac{60 \times 4}{4} = 60 \text{ kNm}$$

Free moment diagram for BC is a symmetric parabola with maximum ordinate

$$= \frac{30 \times 4^2}{8} = 60 \text{ kNm}$$

Bending moment diagram is drawn treating it as positive, if tension develops on dotted side [Figure 1.16(b)], and is as shown in Figure 1.16(c).

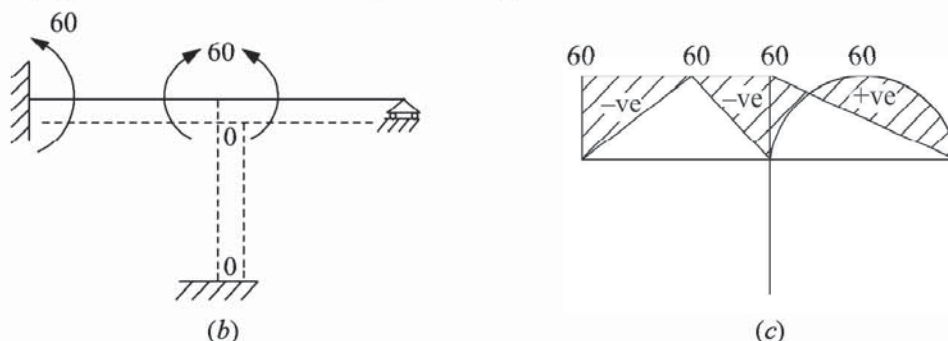


Figure 1.16(b): End moments. **(c)** Bending moment diagram.

Example 1.9 Analyse the frame shown in Figure 1.17(a) by slope deflection method and draw bending moment diagram.

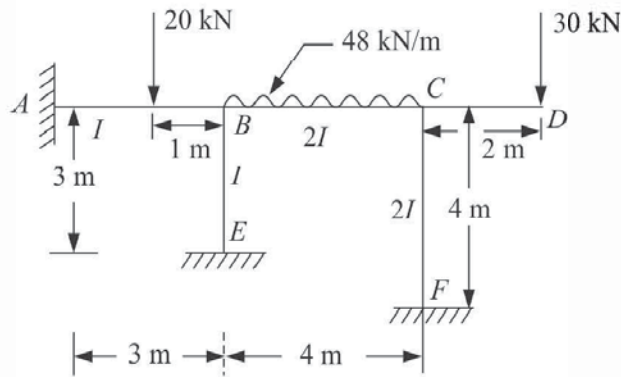


Figure 1.17(a): Given frame.

Solution Fixed End Moments

$$M_{FAB} = -\frac{20 \times 2 \times 1^2}{3^2} = -4.444 \text{ kNm}$$

$$M_{BA} = \frac{20 \times 2^2 \times 1}{3^2} = 8.889 \text{ kNm}$$

$$M_{FBC} = \frac{-48 \times 4^2}{12} = -64 \text{ kNm}$$

$$M_{FCB} = 64 \text{ kNm}$$

$$M_{FCD} = -30 \times 2 = -60 \text{ kNm}$$

$$M_{FBE} = M_{FEB} = M_{FCF} = M_{FFC} = 0$$

Slope Deflection Equations

$$M_{AB} = -4.444 + \frac{2EI}{3} (2\theta_A + \theta_B - 0) = -4.444 + \frac{2}{3} EI \theta_B, \text{ (Since, } \theta_A = 0)$$

$$M_{BA} = 8.889 + \frac{2EI}{3} (\theta_A + 2\theta_B - 0) = 8.889 + \frac{4}{3} EI \theta_B$$

$$M_{BC} = -64 + \frac{2E(2I)}{4} (2\theta_B + \theta_C - 0) = -64 + 2EI \theta_B + EI \theta_C$$

$$M_{CB} = 64 + \frac{2E(2I)}{4} (\theta_B + 2\theta_C - 0) = 64 + EI \theta_B + 2EI \theta_C$$

$$M_{BE} = 0 + \frac{2EI}{3} (2\theta_B + \theta_E) = \frac{4EI}{3} \theta_B, \text{ (Since, } \theta_E = 0)$$

$$M_{EB} = 0 + \frac{2EI}{3} (\theta_B + 2\theta_E) = \frac{2EI}{3} \theta_B, \text{ (Since, } \theta_E = 0)$$

$$M_{CF} = 0 + \frac{2E(2I)}{4}(2\theta_C + \theta_F - 0) = 2EI \theta_C, \quad (\text{Since, } \theta_F = 0)$$

$$M_{FC} = 0 + \frac{2E(2I)}{4}(\theta_C + 2\theta_F - 0) = EI \theta_C, \quad (\text{Since, } \theta_F = 0)$$

Equilibrium Equations

$$\Sigma M_B = 0, \text{ gives}$$

$$M_{BA} + M_{BC} + M_{BE} = 0$$

$$8.889 + \frac{4}{3}EI \theta_B - 64 + 2EI \theta_B + EI \theta_C + \frac{4}{3}EI \theta_B = 0$$

$$4.667 EI \theta_B + EI \theta_C = 55.111 \quad \dots(1.29)$$

$$\Sigma M_C = 0, \text{ gives}$$

$$M_{CB} + M_{CD} + M_{CF} = 0$$

$$64 + EI \theta_B + 2 EI \theta_C - 60 + 2 EI \theta_C = 0$$

$$[\text{Since, } M_{CD} = -60 \text{ kNm, Cantilever moment}]$$

$$EI \theta_B + 4 EI \theta_C = -4 \quad \dots(1.30)$$

Subtracting Eqn. (1.30) from 4 times Eqn. (1.29), we get,

$$(18.668 - 1) EI \theta_B = 220.444 + 4$$

$$EI \theta_B = 12.703$$

Substituting it in Eqn. (1.29), we get,

$$4.667 \times 12.703 + EI \theta_C = 55.111$$

$$\therefore EI \theta_C = -4.174$$

Final Moments

Substituting the values of $EI \theta_B$ and $EI \theta_C$ in slope deflection equations, we get,

$$M_{AB} = -4.444 + \frac{2}{3} \times 12.703 = 4.025 \text{ kNm}$$

$$M_{BA} = 8.889 + \frac{4}{3} \times 12.703 = 25.827 \text{ kNm}$$

$$M_{BC} = -64 + 2 \times 12.703 + (-4.176) = -42.770 \text{ kNm}$$

$$M_{CB} = 64 + 12.703 + 2(-4.176) = -68.351 \text{ kNm}$$

$$M_{BE} = \frac{4}{3} \times 12.703 = 16.937 \text{ kNm}$$

$$M_{EB} = \frac{2}{3} \times 12.703 = 8.469 \text{ kNm}$$

$$M_{CF} = 2(-4.176) = -8.352 \text{ kNm}$$

$$M_{FC} = -4.176 \text{ kNm}$$

Bending Moment Diagram

In the above end moment expression, the sign convention for clockwise is positive and that of anticlockwise is negative. These directions are marked on the sketch of the frame as shown in

Figure 1.17(b). From this it is possible to identify whether and moment causes tension or compression on the dotted side.

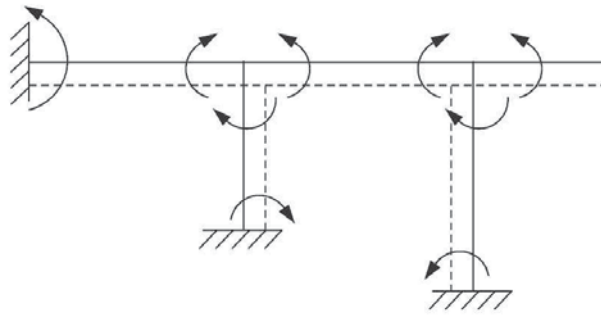


Figure 1.17(b): End moments.

Free moment diagram in beam AB is a triangle with maximum ordinate under load and its magnitude is

$$= \frac{20 \times 2 \times 1}{3} = 13.333 \text{ kNm}$$

Free moment diagram for span BC is a symmetric parabola with maximum ordinate

$$= \frac{48 \times 4^2}{8} = 96 \text{ kNm}$$

These diagrams are drawn on compression side of the member; end moment diagram is drawn on the tension side so that the difference diagram, *i.e.*, the final diagram, can be seen easily. For cantilever portion, tension is at top and bending moment varies from zero at free end to 60 kNm at C . Hence, bending moment diagram is as shown in Figure 1.17(c).

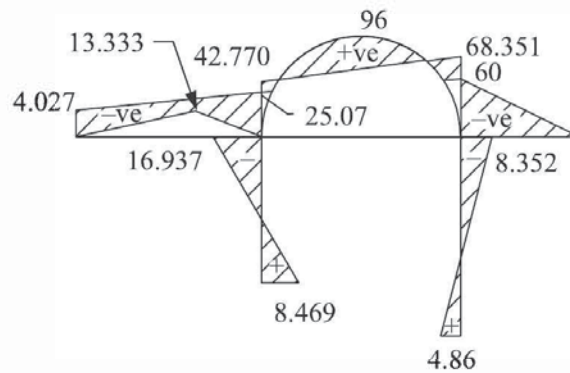


Figure 1.17(c): Bending moment diagram.

Example 1.10 Analyse the frame shown in Figure 1.18(a) by slope deflection method and draw bending moment diagram.

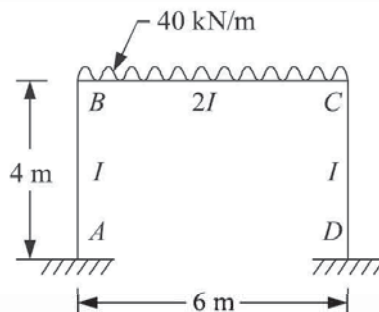


Figure 1.18(a): Given frame.

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Solution Due to symmetry there is no sway in the frame.

Fixed End Moments

$$M_{FAB} = M_{FBC} = M_{FDC} = M_{FCD} = 0$$

$$M_{FBC} = \frac{-40 \times 6^2}{12} = -120 \text{ kNm}$$

$$M_{FCB} = 120 \text{ kNm}$$

Slope Deflection Equations

$$M_{AB} = 0 + \frac{2EI}{4}(2\theta_A + \theta_B - 0) = 0.5 EI \theta_B, \quad (\text{Since, } \theta_A = 0)$$

$$M_{BA} = 0 + \frac{2EI}{4}(\theta_A + 2\theta_B - 0) = EI \theta_B, \quad (\text{Since, } \theta_A = 0)$$

$$M_{BC} = -120 + \frac{2E(2I)}{6}(2\theta_B + \theta_C - 0) = -120 + \frac{4}{3}EI \theta_B + \frac{2}{3}EI \theta_C$$

$$M_{CB} = 120 + \frac{2E(2I)}{6}(\theta_B + 2\theta_C - 0) = 120 + \frac{2}{3}EI \theta_B + \frac{4}{3}EI \theta_C$$

$$M_{CD} = 0 + \frac{2EI}{4}(2\theta_C + \theta_D - 0) = EI \theta_C, \quad (\text{Since, } \theta_D = 0)$$

$$M_{DC} = 0 + \frac{2EI}{4}(\theta_C + 2\theta_D - 0) = 0.5 EI \theta_C, \quad (\text{Since, } \theta_D = 0)$$

Equilibrium Equations

$$\Sigma M_B = 0, \text{ gives}$$

$$M_{BA} + M_{BC} = 0$$

$$EI \theta_B - 120 + \frac{4}{3}EI \theta_B + \frac{2}{3}EI \theta_C = 0$$

Multiplying throughout by 3 and rearranging, we get,

$$7 EI \theta_B + 2 EI \theta_C = 360 \quad \dots(1.31)$$

$$\Sigma M_C = 0, \text{ gives}$$

$$M_{CB} + M_{CD} = 0$$

$$120 + \frac{2}{3}EI \theta_B + \frac{4}{3}EI \theta_C + EI \theta_C = 0$$

Multiplying throughout by 3 and rearranging, we get,

$$2 EI \theta_B + 7 EI \theta_C = -360 \quad \dots(1.32)$$

Multiplying Eqn. (1.31) by 3.5, we get,

$$24.5 EI \theta_B + 7 EI \theta_C = 1260 \quad \dots(1.33)$$

Subtracting Eqn. (1.32) from it, we get,

$$22.5 EI \theta_B = 1620$$

or

$$EI \theta_B = 72$$

Hence, from Eqn. (1.31),

$$7 \times 72 + 2 EI \theta_C = 360$$

$$EI \theta_C = -72$$

Final Moments

Substituting the values of $EI \theta_B$ and $EI \theta_C$ in slope deflection equations, we get,

$$M_{AB} = 0.5 \times 72 = 36 \text{ kNm}$$

$$M_{BA} = 72 \text{ kNm}$$

$$M_{BC} = -120 + \frac{4}{3}(72) + \frac{2}{3}(-72) = -72 \text{ kNm}$$

$$M_{CB} = 120 + \frac{2}{3}(72) + \frac{4}{3}(-72) = 72 \text{ kNm}$$

$$M_{CD} = -72 \text{ kNm}$$

$$M_{DC} = -36 \text{ kNm}$$

Bending Moment Diagram

Noting that the sign convention for clockwise is positive, these moments are marked on the sketch of the frame as shown in Figure 1.18(b). For drawing bending moment diagram, moment creating tension on dotted line side is treated as positive. Free moment diagram in beam BC is a symmetric parabola, with maximum moment

$$= \frac{40 \times 6^2}{8} = 180 \text{ kNm}$$

Bending moment diagram is as shown in Figure 1.18(c).

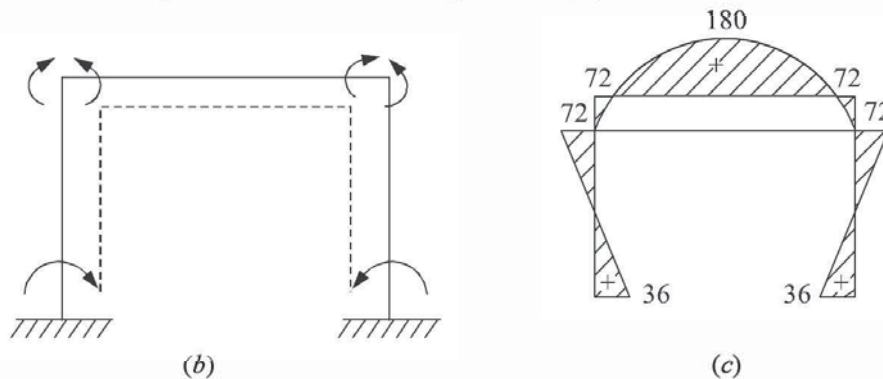


Figure 1.18(b): End moments. (c) Bending moment diagram.

1.5.3 Analysis of Frames with Sway

As discussed in the previous article, there will be a sway in frames, if there is unsymmetry with respect to geometry or loading or due to both. Figure 1.19 shows various one bay one storey frames in which there will be sway. Frame shown in Figure 1.19(a) is having a sway because there is no symmetry of loading as well as symmetry in geometry. Frames shown in Figure 1.19(b) and Figure 1.19(c) are having sway because there is no symmetry with respect to the loading. Frames shown in Figures 1.19(d) and Figure 1.19(e) are having sway because there is no geometric symmetry.

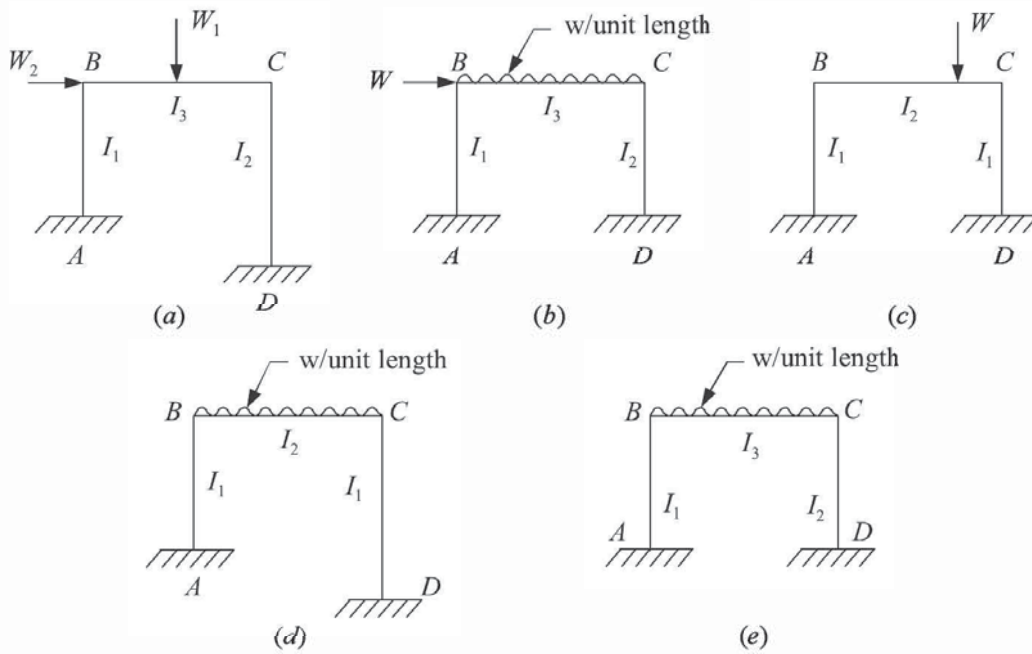
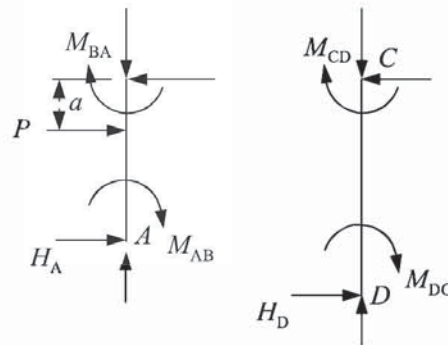
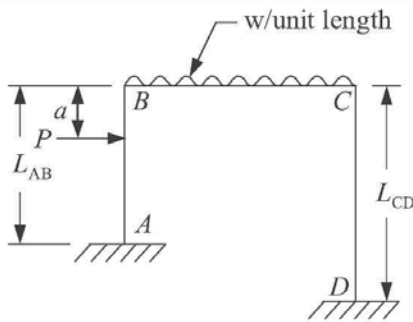


Figure 1.19(a): Sway due to horizontal force and unsymmetry. **(b)** Sway due to horizontal force only. **(c)** Sway due to unsymmetry of load. **(d)** Sway due to unsymmetry of geometry. **(e)** Sway due to unsymmetry of properties of columns.

Consider the analysis of frame shown in Figure 1.20(a). Let Δ be the sway, while writing the slope deflection equations, it may be noted that due to sway the ends are at different levels. Hence, contribution of ‘ Δ ’ terms also should be taken into account. We find in this problem, there are three unknowns, namely θ_B , θ_C and Δ . Moment equilibrium conditions at joints B and C give two equations. One more equation of equilibrium is required. For that, consider the free body diagrams of columns [Refer Figure 1.20(b)], in which the end moments are marked in their positive senses so as to get general expression. Taking moment about top joints, expressions for horizontal shear at supports A and B will be:

$$H_A = \frac{M_{AB} + M_{BA} - P \cdot a}{L_{AB}}$$

$$H_D = \frac{M_{DC} + M_{CD}}{L_{CD}}$$



(a)

(b)

Figure 1.20(a): Typical frame with sway. **(b)** Free body diagrams of columns.

After knowing H_A and H_D , we can write horizontal equilibrium equation for the frame as

$$H_A + H_D + P = 0$$

This gives additional equation required. It is called shear equation/shear condition. Hence, θ_A , θ_B and Δ can be found. The procedure is illustrated with the examples below.

Example 1.11 Analyse the frame shown in Figure 1.21(a) and draw bending moment diagram.

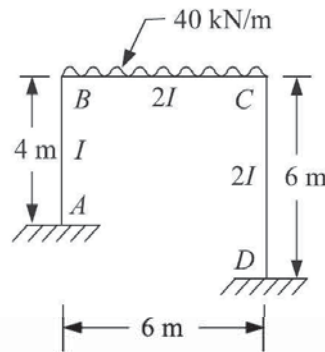


Figure 1.21(a): Given frame.

Solution Let Δ be the sway as shown in Figure 1.21(b). Since, axial deformation is assumed negligible (in member BC) both columns sway by same amount.

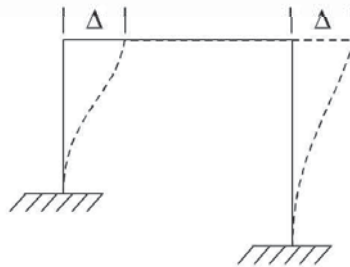


Figure 1.21(b): Frame with sway.

Fixed End Moments

$$M_{FAB} = M_{FBA} = M_{FCD} = M_{FDC} = 0$$

$$M_{FBC} = -\frac{40 \times 6^2}{12} = -120 \text{ kNm}$$

$$M_{FCB} = 120 \text{ kNm}$$

Slope Deflection Equations

$$M_{AB} = 0 + \frac{2EI}{4} \left(2\theta_A + \theta_B - \frac{3\Delta}{4} \right) = 0.5 EI \theta_B - 0.375\Delta, \quad (\text{Since, } \theta_A = 0)$$

$$M_{BA} = 0 + \frac{2EI}{4} \left(\theta_A + 2\theta_B - \frac{3\Delta}{4} \right) = EI \theta_B - 0.375\Delta, \quad (\text{Since, } \theta_A = 0)$$

$$M_{BC} = -120 + \frac{2E(2I)}{6} (2\theta_B + \theta_C - 0) = -120 + 1.333 EI \theta_B + 0.667 EI \theta_C$$

$$M_{CB} = 120 + \frac{2E(2I)}{6} (\theta_B + 2\theta_C - 0) = 120 + 0.667 EI \theta_B + 1.133 EI \theta_C$$

$$M_{CD} = 0 + \frac{2E(2I)}{6} \left(2\theta_C + \theta_D - \frac{3\Delta}{6} \right) = 1.333 EI \theta_C - 0.333 EI \Delta, \quad (\text{Since, } \theta_D = 0)$$

$$M_{DC} = 0 + \frac{2E(2I)}{6} \left(\theta_C + 2\theta_D - \frac{3\Delta}{6} \right) = 0.667 EI \theta_C - 0.333 EI \Delta$$

Equilibrium Equations

$$\Sigma M_B = 0, \text{ gives}$$

$$M_{BA} + M_{BC} = 0$$

$$EI \theta_B - 0.375\Delta - 120 + 1.333 EI \theta_B + 0.667 EI \theta_C = 0$$

$$\text{or } 2.333 EI \theta_B + 0.667 EI \theta_C - 0.375\Delta = 120 \quad \dots(1.34)$$

$$\text{and } \Sigma M_C = 0, \text{ gives}$$

$$M_{CB} + M_{CD} = 0$$

$$120 + 0.667 EI \theta_B + 1.333 EI \theta_C + 1.333 EI \theta_C - 0.333 EI \Delta = 0$$

$$\text{or } 0.667 EI \theta_B + 2.667 EI \theta_C - 0.333\Delta = -120 \quad \dots(1.35)$$

Considering free body diagrams of columns [Refer Figure 1.21(c)] and taking moment about top joints, we get,

$$H_A = \frac{M_{AB} + M_{BA}}{4} \quad \text{and} \quad H_D = \frac{M_{CD} + M_{DC}}{6}$$

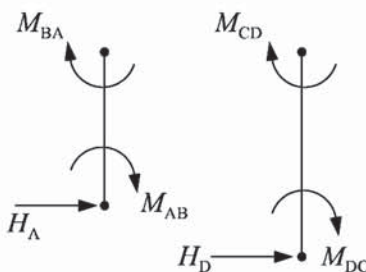


Figure 1.21(c): Free body diagram of columns.

Now considering horizontal equilibrium of the frame, we get,

$$H_A + H_D = 0$$

$$\frac{M_{AB} + M_{BA}}{4} + \frac{M_{CD} + M_{DC}}{6} = 0$$

$$\text{or } 3(M_{AB} + M_{BA}) + 2(M_{CD} + M_{DC}) = 0$$

$$3(0.5 EI \theta_B - 0.375 EI \Delta + EI \theta_B - 0.375 EI \Delta)$$

$$+ 2(1.333 EI \theta_C - 0.333 EI \Delta + 0.667 EI \theta_C - 0.333 EI \Delta) = 0$$

$$\text{i.e., } 4.5 EI \theta_B + 4 EI \theta_C - 3.583 EI \Delta = 0 \quad \dots(1.36)$$

Multiplying Eqn. (1.35) by 3.5, we get,

$$2.333 EI \theta_B + 9.333 EI \theta_C - 1.167 EI \Delta = -420 \quad \dots(1.37)$$

Subtracting Eqn. (1.37) from Eqn. (1.34), we get,

$$-8.667 EI \theta_C + 0.792 EI \Delta = 540 \quad \dots(1.38)$$

Multiplying Eqn. (1.35) by 6.75, we get,

$$4.5 EI \theta_B + 18.00 EI \theta_C - 2.25 \Delta = -810 \quad \dots(1.39)$$

Subtracting Eqn. (1.36) from it, we get,

$$14 EI \theta_C + 1.333 EI \Delta = -810 \quad \dots(1.40)$$

$$\frac{14}{8.667} \text{ time Eqn. (1.38) is}$$

$$-14 EI \theta_C + 1.279 EI \Delta = 872.274 \quad \dots(1.41)$$

Adding Eqn. (1.40) and Eqn. (1.41), we get,

$$2.612 EI \Delta = 62.274$$

or

$$EI \Delta = 23.842$$

Substituting it in Eqn. (1.41), we get,

$$-14 EI \theta_C + 30.493 = 872.274$$

$$EI \theta_C = -60.127$$

Substituting the values of $EI \theta_C$ and $EI \Delta$ in Eqn. (1.36), we get,

$$4.5 EI \theta_B + 4(-60.127) - 3.583 (23.842) = 0$$

$$EI \theta_B = 72.43$$

Note. Calculator may be used to solve three simultaneous equations.

Final Moments

Substituting the values of $EI \theta_B$, $EI \theta_C$, and $EI \Delta$ in slope deflection equations, we get,

$$M_{AB} = 0.5(72.43) - 0.375 \times 23.842 = 27.274 \text{ kNm}$$

$$M_{BA} = 72.43 - 0.375 \times 23.842 = 63.489 \text{ kNm}$$

$$M_{BC} = -120 + 1.333 \times 72.43 + 0.667 (-60.127) = -63.556 \text{ kNm}$$

$$M_{CB} = 120 + 0.667 \times 72.43 + 1.333(-60.127) = 88.162 \text{ kNm}$$

$$M_{CD} = 1.333(-60.127) - 0.333 \times 23.842 = -88.162 \text{ kNm}$$

$$M_{DC} = 0.667(-60.127) - 0.333 \times 23.842 = -48.044 \text{ kNm}$$

Bending Moment Diagram

These moments are first marked on a sketch of frame as shown in Figure 1.21(d). For drawing bending moment diagram tension on the dotted side is taken as positive moment, which is as shown in Figure 1.21(e)

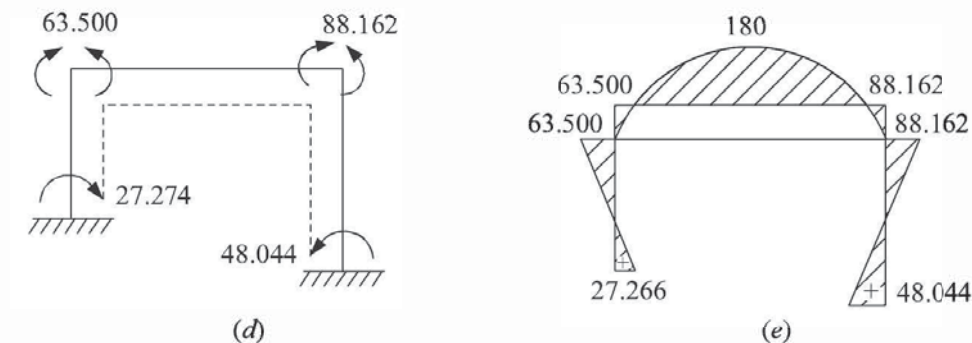


Figure 1.21 (d): End moments. **(e)** Bending moment diagram.

Note: Due to round-off errors $M_{BA} \neq M_{BC}$ and $M_{CB} \neq M_{CD}$. While drawing bending moment diagrams they may be rounded-off and made equal.

Example 1.12 Analyse the frame shown in Figure 1.22(a) by slope deflection method.

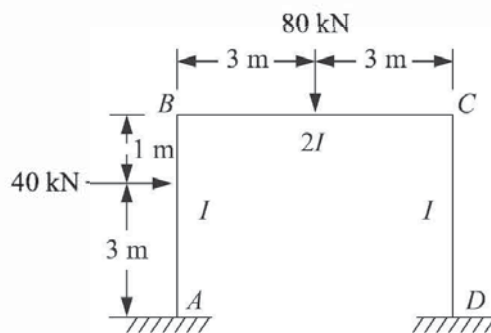


Figure 1.22(a): Given frame.

Solution Let the columns sway by Δ as shown in Figure 1.22(b)

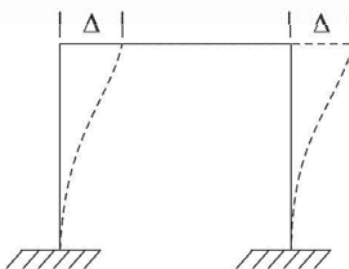


Figure 1.22(b): Frame with sway.

Fixed End Moments

$$M_{FAB} = -\frac{40 \times 3 \times 1^2}{4^2} = -7.5 \text{ kNm}$$

$$M_{FBA} = \frac{40 \times 3^2 \times 1}{4^2} = 22.5 \text{ kNm}$$

$$M_{FBC} = -\frac{80 \times 6}{8} = -60 \text{ kNm}$$

$$M_{FCB} = 60 \text{ kNm}$$

$$M_{FCD} = M_{FDC} = 0$$

Slope Deflection Equations

$$\begin{aligned} M_{AB} &= -7.5 + \frac{2EI}{4} \left(2\theta_A + \theta_B - \frac{3\Delta}{4} \right) \\ &= -7.5 + 0.5 EI \theta_B - 0.375 EI \Delta, \end{aligned} \quad (\text{Since, } \theta_A = 0)$$

$$\begin{aligned} M_{BA} &= 22.5 + \frac{2EI}{4} \left(\theta_A + 2\theta_B - \frac{3\Delta}{4} \right) \\ &= 22.5 + EI \theta_B - 0.375 EI \Delta, \end{aligned} \quad (\text{Since, } \theta_A = 0)$$

$$\begin{aligned} M_{BC} &= -60 + \frac{2E(2I)}{6} (2\theta_B + \theta_C - 0) \\ &= -60 + 1.333 EI \theta_B + 0.667 EI \theta_C \end{aligned}$$

$$M_{CB} = 60 + \frac{2E(2I)}{6}(\theta_B + 2\theta_C - 0) = 60 + 0.667 EI \theta_B + 1.133 EI \theta_C$$

$$\begin{aligned} M_{CD} &= \frac{2EI}{4} \left(2\theta_C + \theta_D - \frac{3\Delta}{4} \right) \\ &= EI \theta_C - 0.375 EI \Delta, \end{aligned} \quad (\text{Since, } \theta_D = 0)$$

$$\begin{aligned} M_{DC} &= \frac{2EI}{4} \left(\theta_C + 2\theta_D - \frac{3\Delta}{4} \right) \\ &= 0.5 EI \theta_C - 0.375 EI \Delta, \end{aligned} \quad (\text{Since, } \theta_D = 0)$$

Equilibrium Equations

$$\Sigma M_B = 0, \text{ gives}$$

$$M_{BA} + M_{BC} = 0$$

$$\begin{aligned} \text{i.e.,} \quad 22.5 + EI \theta_B - 0.375 EI \Delta - 60 + 1.333 EI \theta_B + 0.667 EI \theta_C &= 0 \\ 2.333 EI \theta_B + 0.667 EI \theta_C - 0.375 EI \Delta &= 37.5 \end{aligned} \quad \dots(1.42)$$

and

$$\Sigma M_C = 0, \text{ gives}$$

$$M_{CB} + M_{CD} = 0$$

$$60 + 0.667 EI \theta_B + 1.333 EI \theta_C + EI \theta_C - 0.375 EI \Delta = 0$$

$$\text{i.e.,} \quad 0.667 EI \theta_B + 2.333 EI \theta_C - 0.375 EI \Delta = -60 \quad \dots(1.43)$$

Shear Equations

Consider the free body diagram of columns shown in Figure 1.22(c). Taking moments about top joints,

$$H_A \times 4 + 40 \times 1 = M_{AB} + M_{BA}$$

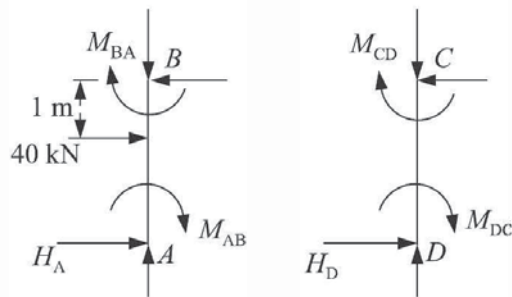


Figure 1.22(c): Free body diagram of columns.

$$\begin{aligned} 4H_A &= M_{AB} + M_{BA} - 40 \times 1 \\ &= -7.5 + 0.5 EI \theta_B - 0.375 EI \Delta + 22.5 + EI \theta_B - 0.375 EI \Delta - 40 \end{aligned}$$

$$\text{i.e.,} \quad 4H_A = 1.5 EI \theta_B - 0.75 EI \Delta - 25$$

$$\begin{aligned} H_D \times 4 &= M_{CD} + M_{DC} \\ &= EI \theta_C - 0.375 EI \Delta + 0.5 EI \theta_C - 0.375 EI \Delta \\ &= 1.5 EI \theta_C - 0.75 EI \Delta \end{aligned}$$

and

$$\Sigma H = 0, \text{ gives}$$

$$H_A + H_D + 40 = 0$$

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$$\begin{aligned} \therefore \quad & 4H_A + 4H_D + 160 = 0 \\ \text{i.e.,} \quad & 1.5 EI \theta_B - 0.75 EI \Delta - 25 + 1.5 EI \theta_C - 0.75 EI \Delta + 160 = 0 \\ & 1.5 EI \theta_B + 1.5 EI \theta_C - 1.5 EI \Delta + 135 = 0 \\ & EI \theta_B + EI \theta_C - EI \Delta = -90 \end{aligned} \quad \dots(1.44)$$

Multiplying Eqn. (1.43) by 3.5, we get,

$$2.333 EI \theta_B + 8.166 EI \theta_C - 1.3125 EI \Delta = -210 \quad \dots(1.45)$$

Subtracting Eqn. (1.45) from Eqn. (1.42), we get,

$$-7.5 EI \theta_C + 0.9375 EI \Delta = 247.5 \quad \dots(1.46)$$

Dividing Eqn. (1.43) by 0.667, we get,

$$EI \theta_B + 3.5 EI \theta_C - 0.5625 EI \Delta = -90 \quad \dots(1.47)$$

Subtracting Eqn. (1.44) from Eqn. (1.47), we get,

$$2.5 EI \theta_C + 0.4375 EI \Delta = 0$$

$$\text{or} \quad 7.5 EI \theta_C + 1.3125 EI \Delta = 0 \quad \dots(1.48)$$

Adding Eqn. (1.46) and Eqn. (1.48), we get,

$$2.25 EI \Delta = 247.5$$

$$EI \Delta = 110$$

\therefore From Eqn. (1.48), $EI \theta_C = -19.25$

Substituting these values of $EI \Delta$ and $EI \theta_C$ in Eqn. (1.44), we get,

$$EI \theta_B + (-19.25) - 110 = -90$$

$$\text{or} \quad EI \theta_B = 39.25$$

Final End Moments

Substituting the values of $EI \theta_B$, $EI \theta_C$, and $EI \Delta$ in slope deflection equations, final moments are obtained.

$$M_{AB} = -7.5 + 0.5(39.25) - 0.375(110) = -29.12 \text{ kNm}$$

$$M_{BA} = 22.5 + 39.25 - 0.375(110) = 20.5 \text{ kNm}$$

$$M_{BC} = -60 + 1.333(39.25) - 0.667(-19.25) = -20.5 \text{ kNm}$$

$$M_{CB} = 60 + 0.667(39.25) + 1.333(-19.25) = 60.51 \text{ kNm}$$

$$M_{CD} = (-19.25) - 0.375 \times 110 = -60.51 \text{ kNm}$$

$$M_{DC} = 0.5(-19.25) - 0.375 \times 110 = -50.875 \text{ kNm}$$

Example 1.13 Analyse the portal frame shown in Figure 1.23(a) by slope deflection method.

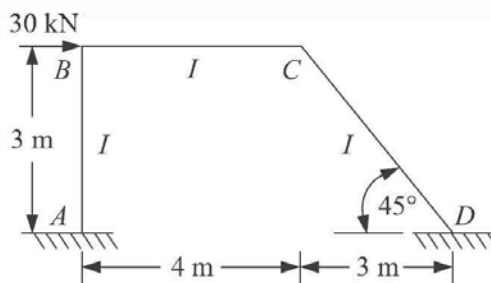


Figure 1.23(a): Portal frame.

Solution Fixed End Moments

In all members, fixed end moment is zero.

Slope Deflection Equations

Figure 1.23(b) shows deflected frame. Let B move to B' and BB' be equal to Δ . Since, axial deformations are neglected, $CC' = \Delta$. However, final position of C cannot be C' because as per the assumption in slope deflection method, member DC cannot have axial deformation. CD can have movement only normal to itself. CC'' is the line normal to CD . As BC can have only normal movement and no axial deformation C'' is taken at normal to CC' at C' . Now $\Delta CC'C''$ represent end settlements of members. Since, CC'' is normal to horizontal and CC'' normal to DC , $\angle CC''C' = \theta = 45^\circ$.

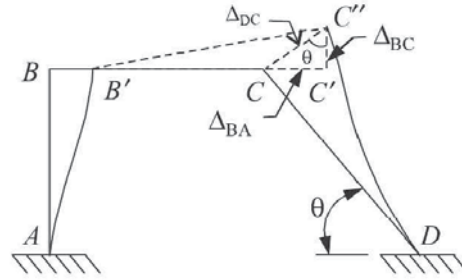


Figure 1.23(b): Deflected shape of frame.

Now, $\Delta_{BA} = \Delta$

$\therefore \Delta_{BC} = \Delta$

and $\Delta_{DC} = CC'' = \frac{\Delta}{\cos \theta} = \sqrt{2} \Delta$

While writing slope deflection equations, relative settlement of ends of members ($\Delta_{AB} = \Delta_{BC} = \Delta_{DC}$) should be carefully considered.

$$\begin{aligned} M_{AB} &= 0 + \frac{2EI}{3} \left(2\theta_A + \theta_B - \frac{3\Delta_{AB}}{3} \right) \\ &= 0.667 EI \theta_B - 0.667 EI \Delta, \end{aligned} \quad (\text{Since, } \theta_A = 0 \text{ and } \Delta_{AB} = \Delta)$$

$$\begin{aligned} M_{BA} &= 0 + \frac{2EI}{3} \left(\theta_A + 2\theta_B - \frac{3\Delta_{AB}}{3} \right) \\ &= 1.333 EI \theta_B - 0.667 EI \Delta \end{aligned} \quad (\text{Since, } \Delta_{AB} = \Delta)$$

$$M_{BC} = 0 + \frac{2EI}{4} \left(2\theta_B + \theta_C - 3 \frac{(\Delta_{AB})}{4} \right)$$

Δ_{AB} is negative, since the position of C is above that of B .

$$M_{BC} = EI \theta_B + 0.5 EI \theta_C + 0.375 EI \Delta$$

$$\begin{aligned} M_{CB} &= 0 + \frac{2EI}{4} \left(\theta_B + 2\theta_C - 3 \frac{(\Delta_{BC})}{4} \right) \\ &= 0.5 EI \theta_B + EI \theta_C + 0.375 EI \Delta \end{aligned} \quad (\text{Since, } \Delta_{AB} = -\Delta)$$

$$M_{CD} = 0 + \frac{2EI}{5} \left(2\theta_C + \theta_D - 3 \frac{(\Delta_{CD})}{5} \right)$$

$$\begin{aligned}
 &= 0.8 EI \theta_C - 0.24 EI(\sqrt{2} \Delta), \quad (\text{Since, } \theta_D = 0 \text{ and } \Delta_{CD} = \sqrt{2} \Delta) \\
 &= 0.8 EI \theta_C - 0.339 EI \Delta \\
 M_{DC} &= 0 + \frac{2EI}{5} \left(\theta_C + 2\theta_D - \frac{3(\sqrt{2}\Delta)}{5} \right) = 0.4 EI \theta_C - 0.339 EI \Delta
 \end{aligned}$$

Equilibrium Equations

$\Sigma M_B = 0$, gives

$$M_{BA} + M_{BC} = 0$$

$$1.333 EI \theta_B - 0.667 EI \Delta + EI \theta_B + 0.5 EI \theta_C + 0.375 EI \Delta = 0$$

i.e., $2.333 EI \theta_B + 0.5 EI \theta_C - 0.292 EI \Delta = 0$... (1.49)

and $\Sigma M_C = 0$, gives

$$M_{CB} + M_{CD} = 0$$

$$0.5 EI \theta_B + EI \theta_C + 0.375 EI \Delta + 0.8 EI \theta_C - 0.339 EI \Delta = 0$$

$$0.5 EI \theta_B + 1.8 EI \theta_C + 0.036 EI \Delta = 0$$
 ... (1.50)

Shear Equations

Consider the free body diagram of columns AB and BC [Figure 1.23(c)]

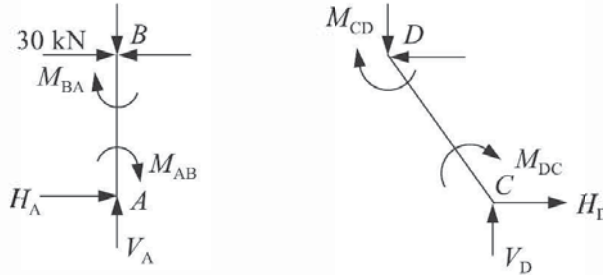


Figure 1.23(c): Free body diagram of columns.

Taking moment about B,

$$H_A \times 3 = M_{AB} + M_{BA} \quad \dots(1.51)$$

Taking moment about D,

$$H_D \times 3 + V_D \times 3 = M_{CD} + M_{DC} \quad \dots(1.52)$$

To find V_D , moment about B in entire frame may be considered

$$V_D \times 7 = 30 \times 3$$

or $V_D = 12.857 \text{ kN}$

$$\therefore H_D \times 3 + 3 \times 12.857 = M_{CD} + M_{DC}$$

and $\Sigma H = 0$ for entire frame gives

$$H_A + H_D + 30 = 0$$

$$3H_A + 3H_D + 90 = 0$$

$$M_{AB} + M_{BA} + M_{CD} + M_{DC} - 38.571 + 90 = 0$$

$$0.667 EI \theta_B - 0.667 EI \Delta + 1.333 EI \theta_B + 0.8 EI \theta_C - 0.339 EI \Delta$$

$$+ 0.4 EI \theta_C + 0.339 EI \Delta = - 51.429$$

$$\text{i.e.,} \quad 2 EI \theta_B + 1.2 EI \theta_C - 2.011 EI \Delta = -51.429 \quad \dots(1.53)$$

Dividing Eqn. (1.51) by $\frac{2.333}{2}$, we get,

$$2 EI \theta_B + 0.429 EI \theta_C - 0.250 EI \Delta = 0 \quad \dots(1.54)$$

Subtracting Eqn. (1.54) from Eqn. (1.53), we get,

$$0.771 EI \theta_C - 1.761 EI \Delta = -51.429 \quad \dots(1.55)$$

Multiplying Eqn. (1.52) by 4, we get,

$$2 EI \theta_B + 7.2 EI \theta_C + 0.144 EI \Delta = 0 \quad \dots(1.56)$$

Subtracting Eqn. (1.53) from Eqn. (1.56), we get,

$$6 EI \theta_C + 2.155 EI \Delta = 51.429 \quad \dots(1.57)$$

Multiplying Eqn. (1.55) by $\frac{6}{0.771}$, we get,

$$6 EI \theta_C - 13.704 EI \Delta = -400.225 \quad \dots(1.58)$$

Subtracting Eqn. (1.58) from Eqn. (1.57), we get,

$$15.859 EI \Delta = 451.655$$

$$\text{or} \quad \Delta = 28.480$$

From Eqn. (1.57),

$$6 EI \theta_C + 2.155 \times 28.480 = 51.429$$

$$EI \theta_C = -1.658$$

Substituting these values of $EI \Delta$ and $EI \theta_C$ in Eqn. (1.53), we get,

$$2 EI \theta_B + 1.2(-1.658) - 2.011(28.480) = -51.429$$

$$EI \theta_B = 3.917$$

Note: Calculator may be used to solve three simultaneous equations.

End Moments

$$M_{AB} = 0.667(3.917) - 0.667(28.480) = -16.384 \text{ kNm}$$

$$M_{BA} = 1.333(3.917) - 0.667(28.480) = -13.770 \text{ kNm}$$

$$M_{BC} = 3.917 + 0.5(-1.658) + 0.375 \times 28.480 = 13.770 \text{ kNm}$$

$$M_{CB} = 0.5(3.917) - 1.658 + 0.375 \times 28.480 = 10.981 \text{ kNm}$$

$$M_{CD} = 0.8(-1.658) - 0.339 \times 28.480 = -10.981 \text{ kNm}$$

$$M_{DC} = 0.4(-1.658) - 0.339 \times 28.480 = -16.286 \text{ kNm}$$

Note: For solving simultaneous equations up to 3rd order, calculator may be used.

SUMMARY

1. Assuming joints as rigid and neglecting axial and shear deformations, the slope-deflection equation is derived as:

$$M_{AB} = M_{FAB} + \frac{2EI}{L} \left(2\theta_A + \theta_B - \frac{3\Delta}{L} \right)$$

In the above expression, clockwise moment and rotations are positive and settlement of right hand side support over left hand support is positive.

2. Considering each joint as fixed, the fixed end moments are found and slope-deflection equations are written in terms of unknown displacements. Then joint equilibrium equations are formed and solved to get unknown displacements.
3. Then joint moments in each member are found and *SF* and *BM* diagrams drawn.
4. Procedure is illustrated by taking the analysis of continuous beams, frames without and with sway.

MULTIPLE CHOICE QUESTIONS

Select the correct option:

1. The final moment at the end *A* in a beam *AB* due to rotations θ_A , θ_B , and downward settlement of support at *B* is given by:

(a) $M_{FAB} + \frac{4EI}{L^2} \left(\theta_A + 2\theta_B - \frac{3\Delta}{L} \right)$

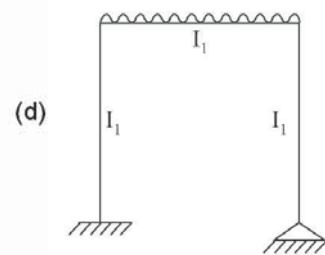
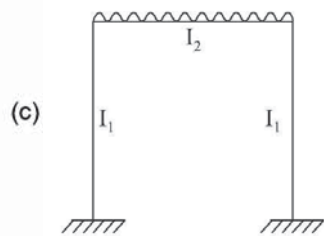
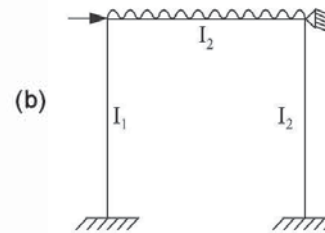
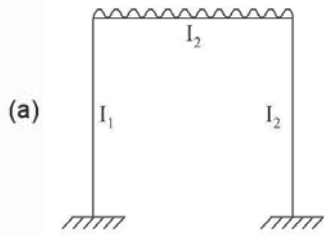
(b) $M_{FAB} + \frac{2EI}{L^2} \left(2\theta_A + \theta_B - \frac{3\Delta}{L} \right)$

(c) $M_{FAB} + \frac{2EI}{L^2} \left(\theta_A + 2\theta_B - \frac{3\Delta}{L} \right)$

(d) $M_{FAB} + \frac{4EI}{L^2} \left(\theta_A + 2\theta_B + \frac{3\Delta}{L} \right)$

[Ans: 1. (c)]

2. Which one of the frames shown in the figure shown below does not sway?



[Ans: 2. (b)]

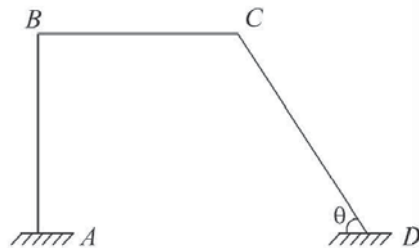
3. In the frame shown in the figure, if the lateral sway of *BC* is Δ , the sway in member *DC* is:

(a) Δ

(b) $\Delta \cos \theta$

(c) $\Delta \sin \theta$

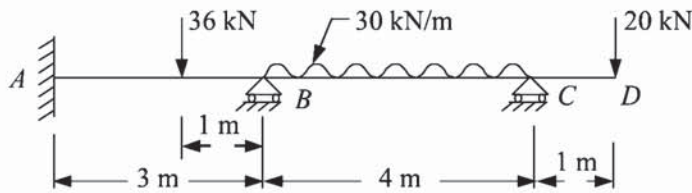
(d) $\Delta \sec \theta$



[Ans: 3. (d)]

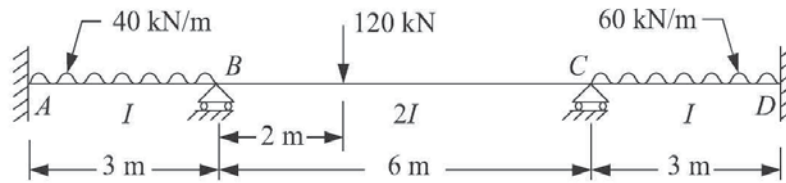
EXERCISES

1. Analyse the continuous beam shown in figure given below by slope deflection method and draw bending moment diagram. Flexural rigidity is constant throughout.



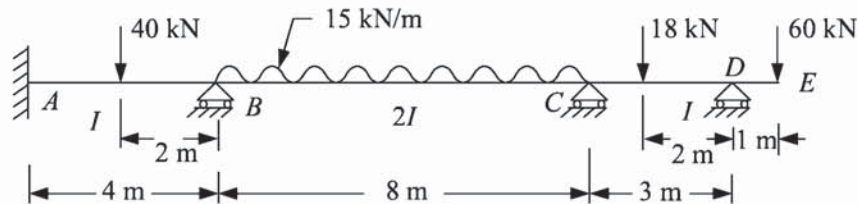
$$\left[\begin{array}{l} \text{Ans: } M_{AB} = 2.88 \text{ kNm;} \\ M_{BA} = 37.76 \text{ kNm} = -M_{BC} \\ M_{CB} = 20 \text{ kNm} \end{array} \right]$$

2. Analyse the continuous beam ABCD shown in figure given below, if support C sinks by 10 mm. Take $E = 2 \times 10^5 \text{ N/mm}^2$ and $I = 4 \times 10^7 \text{ mm}^4$



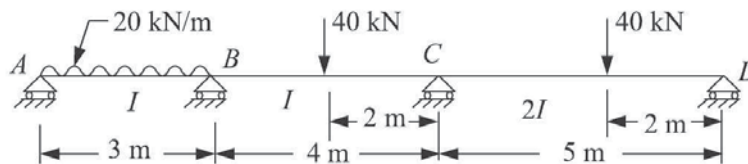
$$\left[\begin{array}{l} \text{Ans: } M_{AB} = -0.111 \text{ kNm;} \\ M_{BA} = -M_{BC} = 89.777 \text{ kNm} \\ M_{CB} = -M_{CD} = 24.111 \text{ kNm} \\ M_{DC} = 82.111 \text{ kNm} \end{array} \right]$$

3. Analyse the continuous beam shown in figure given below by slope deflection method and draw bending moment diagram.



$$\left[\begin{array}{l} \text{Ans: } M_{AB} = 2.67 \text{ kNm;} \\ M_{BA} = -M_{BC} = 65.33 \text{ kNm} \\ M_{CB} = -M_{CD} = 41.33 \text{ kNm} \end{array} \right]$$

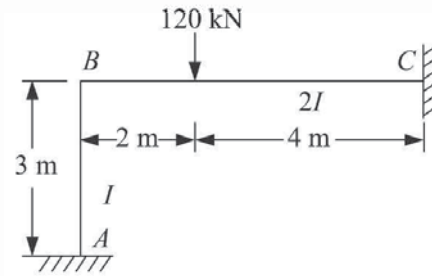
4. Formulate the required equilibrium equations for analysing continuous beam shown in figure given below by slope deflection method.



$$\text{Ans: } \begin{bmatrix} 1.333 & 0.667 & 0 & 0 \\ 0.667 & 2.333 & 0.5 & 0 \\ 0 & 0.5 & 2.6 & 0.8 \\ 0 & 0 & 0.8 & 1.6 \end{bmatrix} \begin{bmatrix} EI \theta_A \\ EI \theta_B \\ EI \theta_C \\ EI \theta_D \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \\ -0.8 \\ -28.8 \end{bmatrix}$$

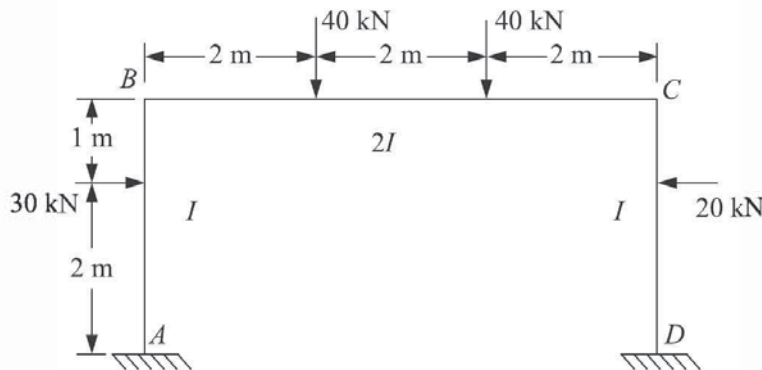
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5. Analyse the frame shown in figure given below by slope deflection method and draw bending moment diagram.



$$\left[\begin{array}{l} \text{Ans: } M_{AB} = 26.667 \text{ kNm;} \\ M_{BA} = -M_{CB} = 53.333 \text{ kNm} \\ M_{CB} = 80 \text{ kNm} \end{array} \right]$$

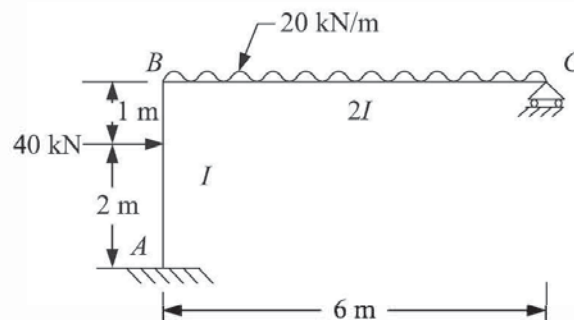
6. Analyse the symmetric frame shown in figure given below by slope deflection method.



$$\left[\begin{array}{l} \text{Ans: } M_{AB} = 6.667 \text{ kNm;} \\ M_{BA} = -M_{BC} = 40 \text{ kNm} \\ M_{CB} = -M_{CD} = 40 \text{ kNm} \\ M_{DC} = -6.67 \text{ kNm} \end{array} \right]$$

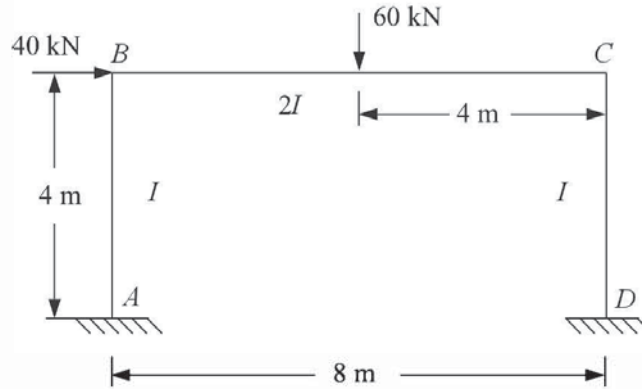
7. Analyse the portal frame shown in figure given below by slope deflection method and draw bending moment diagram.

Note: Unknowns are θ_B , θ_C and Δ



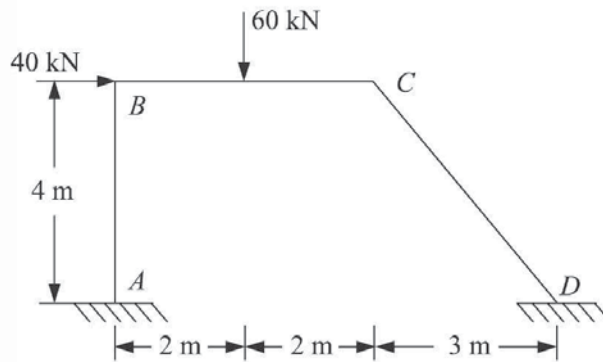
$$\left[\begin{array}{l} \text{Ans: } M_{AB} = -82.5 \text{ kNm;} \\ M_{BA} = -M_{BC} = 2.5 \text{ kNm} \\ M_{CB} = 0 \end{array} \right]$$

8. Analyse the portal frame shown in figure given below by slope deflection method.



$$\left[\begin{array}{l} \text{Ans: } M_{AB} = -25.71 \text{ kNm;} \\ M_{BA} = -M_{BC} = 5.72 \text{ kNm} \\ M_{CB} = -M_{CD} = 74.29 \text{ kNm} \\ M_{DC} = -65.72 \text{ kNm} \end{array} \right]$$

9. Analyse the frame shown in figure given below by slope deflection method. Assume uniform flexural rigidity throughout.



$$\left[\begin{array}{l} \text{Ans: } M_{AB} = -34.64 \text{ kNm;} \\ M_{BA} = -M_{BC} = -21.30 \text{ kNm} \\ M_{CB} = -M_{CD} = 56.59 \text{ kNm} \\ M_{DC} = -47.49 \text{ kNm} \end{array} \right]$$

REVIEW QUESTIONS

1. State the assumptions made in deriving slope-deflection equations.
2. State and derive slope-deflection equation.