15.8. CONTINUOUS BEAMS

Continuous beam is a beam which is supported on more than two supports. Fig. 15.16 shows such a beam, which is subjected to some external loading (here a uniformly distributed load). The deflection curve for the beam is shown by dotted line. The deflection curve is having convexity upwards over the intermediate supports, and concavity upwards over the mid of the span. Hence there will be hogging moments (*i.e.*, negative) over the intermediate supports and sagging moments (*i.e.*, positive) over the mid of the span. The end supports of a simply supported continuous beam will not be subjected to any bending moment. But the end support of fixed continuous beam will be subjected to fixing moments. If the moments over the intermediate supports are known, then the B.M. diagram can be drawn.

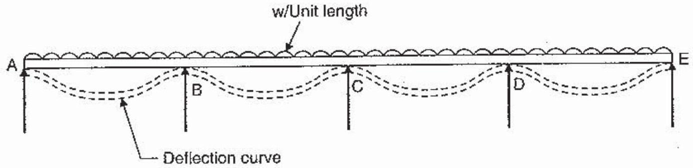


Fig. 15.16

The Fig. 15.16 shows a simply supported continuous beam. In this figure the end supports at A and E will not be subjected to any bending moment. Hence in this case $M_A = M_E = 0$.

Fig. 15.16 (a) shows a continuous beam with fixed ends at A and E. Here the end supports at A and E will be subjected to fixing moments. Hence M_A and M_E will not be zero.

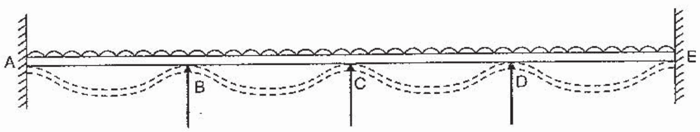


Fig. 15.16 (a)

15.9. BENDING MOMENT DIAGRAM FOR CONTINUOUS BEAMS

In Art. 15.8 it is mentioned that if the moments over the intermediate supports of a continuous beam are known, then the B.M. diagram can be drawn easily. The moments over the intermediate supports are determined by using *Clapeyron's theorem of three moments* which states that :

If BC and CD are any two consecutive span of a continuous beam subjected to an external loading, then the moments M_B, M_C and M_D at the supports B, C and D are given by,

$$M_B.L_1 + 2M_C(L_1 + L_2) + M_D.L_2 = \frac{6a_1\bar{x}_1}{L_1} + \frac{6a_2\bar{x}_2}{L_2} \quad ...(15.12)$$

where L_1 = Length of span BC

L_2 = Length of span CD

a_1 = Area of B.M. diagram due to vertical loads on span BC

a_2 = Area of B.M. diagram due to vertical loads on span CD

$\bar{x}_1$ = Distance of C.G. of the B.M. diagram due to vertical loads on BC from B

$\bar{x}_2$ = Distance of C.G. of the B.M. diagram due to vertical loads on CD from D.

The equation (15.12) is known as the *equation of three moments* or Clapeyron's equation.

15.9.1. Derivation of Clapeyron's Equation of three Moments. Fig. 15.17 shows the length BCD (two consecutive spans) of a continuous beam which is shown in Fig. 15.16. Let M_B, M_C and M_D are the support moments at B, C and D respectively.

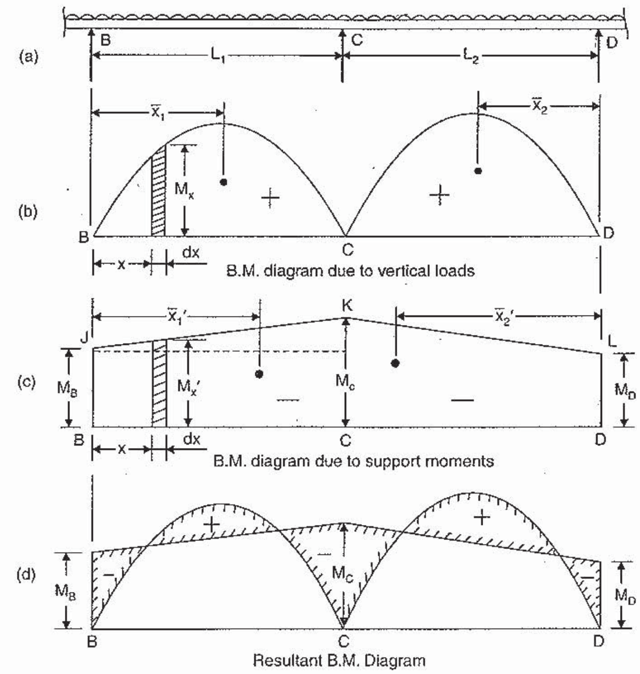


Fig. 15.17

Let L_1 = Length of span BC

L_2 = Length of span CD

a_1 = Area of B.M. diagram due to vertical loads on span BC

a_2 = Area of B.M. diagram due to vertical loads on span CD

a_1' = Area of B.M. diagram due to support moments M_B and M_C

a_2' = Area of B.M. diagram due to support moments M_C and M_D

$\bar{x}_1$ = Distance of C.G. of B.M. diagram due to vertical loads on BC

$\bar{x}_2$ = Distance of C.G. of B.M. diagram due to vertical loads on CD

$\bar{x}_1'$ = Distance of C.G. of B.M. diagram due to support moments on BC

$\bar{x}_2'$ = Distance of C.G. of B.M. diagram due to support moments on CD.

Fig. 15.17 (*b*) and (*c*) shows the B.M. diagrams due to vertical loads and due to supports moments respectively.

(*i*) *Consider the span BC*

Let M_x = B.M. due to vertical loads at a distance x from B (sagging)

M_x' = B.M. due to support moments at a distance x from B (hogging)

∴ Net B.M. at a distance x from B is given by,

$$EI\,\frac{d^2y}{dx^2} = M_x - M_x'$$

Multiplying by x to both sides, we get

$$EI.x.\,\frac{d^2y}{dx^2} = x.M_x - x.M'x$$

Integrating from zero to L_1, we get

$$\int_0^{L_1} EI.x.\frac{d^2y}{dx^2}.dx = \int_0^{L_1} x.M_x.dx - \int_0^{L_1} x.M_x'.dx$$

or $$EI\left[x\,\frac{dy}{dx} - y\right]_0^{L_1} = a_1\bar{x}_1 - a_1'\bar{x}_1' \quad ...(i)$$

(∵ $M_x.dx$ = Area of B.M. diagram of length dx. And $x.M_x.dx$ = Moment of area of B.M. diagram of length dx about B.

Hence $\int_0^{L_1} x.M_x.dx = a_1\bar{x}_1$. And so on)

Substituting the limits in L.H.S. of equation (*i*), we have

$$EI\left[\left\{L_1\left(\frac{dy}{dx}\right)_{\text{at } C} - y_C\right\} - \left\{0 \times \left(\frac{dy}{dx}\right)_{\text{at } B} - y_B\right\}\right]$$

$$= a_1\bar{x}_1 - a_1'\bar{x}_1'$$

or $$EI[(L_1.\theta_C - y_C) - (0 - y_B)] = a_1\bar{x}_1 - a_1'\bar{x}_1'. \quad \left[\because \left(\frac{dy}{dx}\right)_{\text{at } C} = \theta_C\right]$$

But deflection at B and C are zero. Hence $y_B = 0$ and $y_C = 0$. Hence above equation becomes as

$$[EI.L_1.\theta_C = a_1\bar{x}_1 - a_1'\bar{x}_1' \quad ...(ii)$$

But a_1' = Area of B.M. diagram due to supports moments

= Area of trapezium $BCKJ$

$= \frac{1}{2}(M_B + M_C) \times L_1$

and $\bar{x}_1'$ = Distance of C.G. of area $BCKJ$ from B

$$= \frac{M_B.L_1.\frac{L_1}{2} + \frac{1}{2} \times (M_C - M_B).L_1 \times \frac{2L_1}{3}}{M_B.L_1 + \frac{1}{2}(M_C - M_B).L_1}$$

$$= \frac{M_B.\frac{L_1}{2} + (M_C - M_B) \times \frac{L_1}{3}}{M_B + (M_C - M_B).\frac{1}{2}} = \frac{\frac{3M_BL_1 + 2L_1(M_C - M_B)}{6}}{\frac{2M_B + M_C - M_B}{2}}$$

$$= \frac{\frac{L_1}{3}[3M_B + 2M_C - 2M_B]}{M_B + M_C} = \left(\frac{M_B + 2M_C}{M_B + M_C}\right) \times \frac{L_1}{3}$$

Substituting the values of $\bar{a}_1$ and $\bar{x}_1'$ in equation (*ii*), we get

$$EI.L_1.\theta_C = a_1\bar{x}_1 - \frac{1}{2}(M_B + M_C).L_1 \times \left(\frac{M_B + 2M_C}{M_B + M_C}\right) \times \frac{L_1}{3}$$

$$= a_1\bar{x}_1 - \frac{L_1^2}{6}(M_B + 2M_C)$$

or $$6EI.\theta_C = \frac{6a_1\bar{x}_1}{L_1} - L_1(M_B + 2M_C) \quad ...(iii)$$

(*ii*) *Consider the span CD*

Similarly considering the span CD and taking D as origin and x positive to the left, it can be shown that

$$6EI.(-\theta_C) = \frac{6a_2\bar{x}_2}{L_2} - L_2(M_D + 2M_C)$$

[In the above case the slope at C (*i.e.*, θ_C) will have opposite sign than that given by equation (*iii*). The reason is that the direction of x from B for the span BC, and from D for span CD are in the opposite direction].

Hence the above equation becomes as

∴ $$-6EI\theta_c = \frac{6a_2\bar{x}_2}{L_2} - L_2(M_D + 2M_C) \quad ...(iv)$$

Adding equation (*iii*) and (*iv*), we get

$$0 = \frac{6a_1\bar{x}_1}{L_1} - L_1(M_B + 2M_C) + \frac{6a_2\bar{x}_2}{L_2} - L_2(M_D + 2M_C)$$

$$= \frac{6a_1\bar{x}_1}{L_1} + \frac{6a_2\bar{x}_2}{L_2} - L_1M_B - 2L_1M_C - L_2M_D - 2L_2M_C$$

or $$L_1.M_B + L_2M_D + 2M_C(L_1 + L_2) = \frac{6a_1\bar{x}_1}{L_1} + \frac{6a_2\bar{x}_2}{L_2}$$

or $$M_BL_1 + 2M_C(L_1 + L_2) + M_DL_2 = \frac{6a_1\bar{x}_1}{L_1} + \frac{6a_2\bar{x}_2}{L_2}$$

15.9.2. Application of Clapeyron's equation of Three Moments to Continuous Beam with Simply Supported ends. The fixing moments on the ends of a simply supported beam is zero. The continuous beam with simply supported ends may carry uniformly distributed load or point loads as given in the following problems:

Problem 15.9. *A continuous beam ABC covers two consecutive span AB and BC of lengths 4 m and 6 m, carrying uniformly distributed loads of 6 kN/m and 10 kN/m respectively. If the ends A and C are simply supported, find the support moments at A, B and C. Draw also B.M. and S.F. diagrams.*

Sol. Given :

Length AB, $L_1 = 4$ m

Length BC, $L_2 = 6$ m

U.d.l. on AB, $w_1 = 6$ kN/m

U.d.l. on BC, $w_2 = 10$ kN/m

Since the ends A and C are simply supported, the support moments at A and C will be zero.

$$\therefore \quad M_A = M_C = 0$$

To find the support moment at B (*i.e.*, M_B), Clapeyron's equation of three moments should be applied. Hence, we get

$$M_A.L_1 + 2M_B(L_1 + L_2) + M_C.L_2 = \frac{6a_1\bar{x}_1}{L_1} + \frac{6a_2\bar{x}_2}{L_2}$$

or

$$0 \times 4 + 2M_B(4 + 6) + 0 \times L_2 = \frac{6a_1\bar{x}_2}{4} + \frac{6a_2\bar{x}_2}{6}$$

or

$$20M_B = \frac{3a_1\bar{x}_1}{2} + a_2\bar{x}_2 \quad \ldots(i)$$

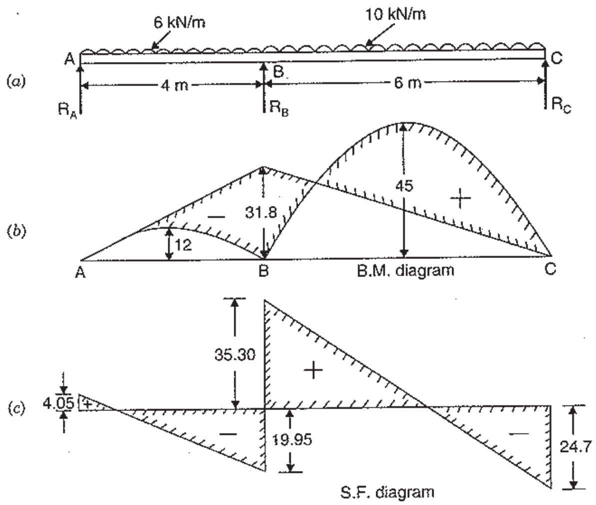


Fig. 15.18

The B.M. diagram on a simply supported beam carrying u.d.l. is a parabola having an altitude of $\frac{wL^2}{8}$. And area of B.M. diagram $= \frac{2}{3} \times$ Span $\times$ Altitude. The distance of C.G. of this area from one end $= \frac{\text{Span}}{2}$.

Now $\quad a_1$ = Area of B.M. diagram due to u.d.l. on AB

$$= \frac{2}{3} \times AB \times \text{Altitude} = \frac{2}{3} \times AB \times \frac{w_1 L_1^2}{8}$$

$$= \frac{2}{3} \times 4 \times \frac{6 \times 4^2}{8} = 32$$

$$\bar{x}_1 = \frac{L_1}{2} = \frac{4}{2} = 2 \text{ m}$$

a_2 = Area of B.M. diagram due to u.d.l. on BC

$$= \frac{2}{3} \times BC \times \frac{w_2 L_2^2}{8} = \frac{2}{3} \times 6 \times \frac{10 \times 6^2}{8} = 180$$

and

$$\bar{x}_2 = \frac{L_2}{2} = \frac{6}{2} = 3 \text{ m.}$$

Substituting these values in equation (i), we get

$$20M_B = \frac{3 \times 32 \times 2}{2} + 180 \times 3$$

$$= 96 + 540 = 636$$

$$\therefore \quad M_B = \frac{636}{20} = 31.8 \text{ kNm.}$$

Now B.M. diagram due to supports moments is drawn as shown in Fig. 15.18 (b) in which

$$M_A = 0,\ M_C = 0 \text{ and } M_B = 31.8 \text{ kNm.}$$

The B.M. diagram due to vertical loads (here u.d.l.) on span AB and span BC are also shown by parabolas of altitudes

$$\frac{w_1 L_1^2}{8} = \frac{6 \times 4^2}{8} = 12 \text{ kNm and } \frac{w_2 L_2^2}{8} = \frac{10 \times 6^2}{8} = 45 \text{ kNm}$$

respectively in Fig. 15.18 (b).

S.F. Diagram

First calculate the reactions R_A, R_B and R_C at A, B and C respectively. For the span AB, taking moments about B, we get

$$R_A \times 4 - 6 \times 4 \times \frac{4}{2} = M_B \quad \text{(The support } B \text{ has moment } M_B\text{)}$$

$$= -31.8$$

($\because$ $M_B = 31.8$. Negative sign is taken as the moment at B is hogging)

or

$$4R_A - 48 = -31.8$$

or

$$R_A = \frac{-31.8 + 48}{4} = 4.05 \text{ kN.}$$

Similarly for the span BC, taking moments about B, we get

$$R_C \times 6 - 6 \times 10 \times \frac{6}{2} = M_B = -31.8$$

or

$$6R_C - 180 = -31.8$$

or

$$R_C = \frac{180 - 31.8}{6} = 24.7 \text{ kN.}$$

Now $\quad R_B$ = Total load on $ABC - (R_A + R_C)$

$$= (6 \times 4 + 10 \times 6) - (4.05 + 24.7) = 55.25 \text{ kN.}$$

Now complete the S.F. diagram as shown in Fig. 15.18 (c).

Problem 15.10. *A continuous beam ABCD of length 15 m rests on four supports covering 3 equal spans and carries a uniformly distributed load of 1.5 kN/m length. Calculate the moments and reactions at the supports. Draw the S.F. and B.M. diagrams also.*

Sol. Given :

Length AB, $L_1 = 5$ m

Length BC, $L_2 = 5$ m

Length CD, $L_3 = 5$ m

U.d.l., $w_1 = w_2 = w_3 = 1.5$ kN/m.

Since ends A and D are simply supported, the support moments at A and D will be zero.

$\therefore$ $M_A = 0$ and $M_D = 0$

From symmetry $M_B = M_C$

To find the support moments at B and D, Clapeyron's equation of three moments is applied for ABC and for BCD.

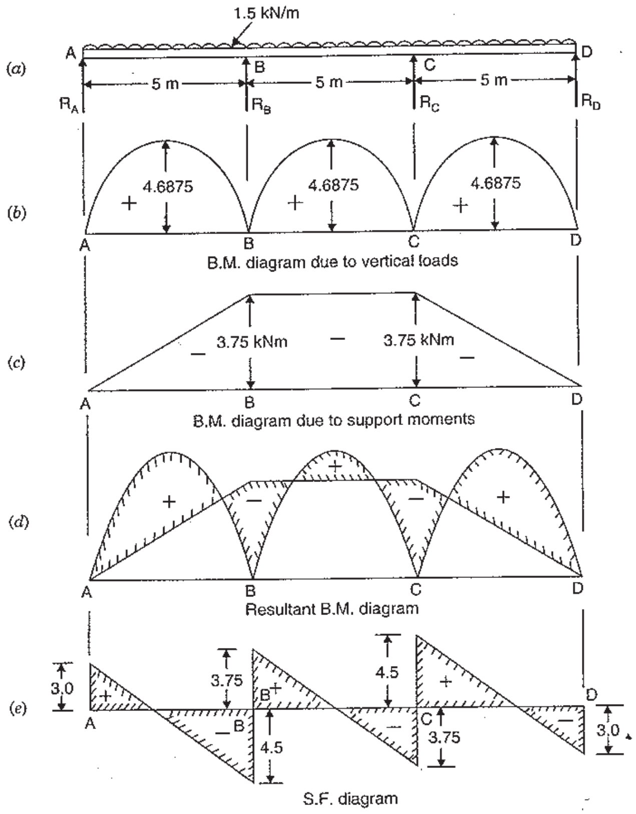


Fig. 15.19

For ABC, we get

$$M_A L_1 + 2M_B(L_1 + L_2) + M_C \cdot L_2 = \frac{6a_1\bar{x}_1}{L_1} + \frac{6a_2\bar{x}_2}{L_2}$$

or
$$0 \times 5 + 2M_B(5 + 5) + M_C \times 5 = \frac{6a_1\bar{x}_1}{5} + \frac{6a_2\bar{x}_2}{5}$$

or
$$20M_B + 5M_C = \frac{6}{5}(a_1\bar{x}_1 + a_2\bar{x}_2) \quad \ldots(i)$$

Now a_1 = Area of B.M. Diagram due to u.d.l. on AB when AB is considered as simply supported beam

$$= \frac{2}{3} \times AB \times \text{Altitude of parabola}$$

$$= \frac{2}{3} \times 5 \times \frac{w_1 L_1}{8} = \frac{2}{3} \times 5 \times \frac{1.5 \times 5^2}{8} = 15.625$$

$$\bar{x}_2 = \frac{L_1}{2} = \frac{5}{2} = 2.5 \text{ m}$$

Due to symmetry $a_2 = a_1 = 15.625$ and $\bar{x}_2 = \bar{x}_1 = 2.5$

Substituting these values in equation (*i*), we get

$$20M_B + 5M_C = \frac{6}{5}(15.625 \times 2.5 + 15.625 \times 2.5)$$

$$= \frac{6}{5} \times 2 \times 15.625 \times 2.5 = 93.750$$

or
$$20M_B + 5M_B = 93.750 \quad (\because M_B = M_C \text{ due to symmetry})$$

or
$$M_B = \frac{93.750}{25} = 3.75 \text{ kNm}$$

$\therefore$ $M_B = M_C =$ **3.75 kNm. Ans.**

Now the B.M. diagram due to supports moments is drawn as shown in Fig. 15.19 (*c*), in which

$$M_A = 0, M_D = 0, M_B = M_C = 3.75 \text{ kNm}.$$

The B.M. diagram due to vertical loads (here u.d.l.) on span AB, BC and CD (considering each span as simply supported) are shown by parabolas of altitudes $\frac{w_1 L_1^2}{8} = \frac{1.5 \times 5^2}{8} = 4.6875$ kNm each in Fig. 15.19 (*b*). Resultant B.M. diagram is shown in Fig. 15.19 (*d*).

Support Reactions

Let R_A, R_B, R_C and R_D are the support reactions at A, B, C and D respectively.

Due to symmetry, $R_A = R_D$

$R_B = R_C$

For the span AB, taking moments about B, we get

$$M_B = R_A \times 5 - 1.5 \times 5 \times \frac{5}{2}$$

or
$$-3.75 = R_A \times 5 - 18.75 \quad (\because M_B = -3.75)$$

or
$$5R_A = 18.75 - 3.75 = 15$$

$\therefore$ $R_A = \frac{15}{5} =$ **3.0 kN. Ans.**

$\therefore$ Due to symmetry, $R_D = R_A =$ **3.0 kN. Ans.**

Now $R_A + R_B + R_C + R_D =$ Total load on $ABCD$

or $R_A + R_B + R_B + R_A = 1.5 \times 15$ $\quad (\because R_C = R_B, R_D = R_A)$

or $2(R_A + R_B) = 22.5$

or $R_A + R_B = \frac{22.5}{2} = 11.25$

or $R_B = 11.25 - R_A = 11.25 - 3.00 = 8.25$ $\quad (\because R_A = 3.0)$

$\therefore$ $R_B = R_C =$ **8.25 kN. Ans.**

Now the S.F. diagram can be drawn as shown in Fig. 15.19 (*e*).

Problem 15.11. *A continuous beam ABCD, simply supported at A, B, C and D is loaded as shown in Fig. 15.20 (a). Find the moments over the beam and draw B.M. and S.F. diagrams.*

Sol. Given :

Length AB, $L_1 = 6$ m

Length BC, $L_2 = 5$ m

Length CD, $L_3 = 4$ m

Point load in BD, $W_1 = 9$ kN

Point load in BC, $W_2 = 8$ kN

U.d.l. on CD, $w = 3$ kN/m.

(*i*) *B.M. diagram due to vertical loads taking each span as simply supported*

Consider beam AB as simply supported

B.M. at point load at $E = \frac{W_1 \times a \times b}{L_1} = \frac{9 \times 2 \times 4}{6}$ $\quad (\because$ Here $a = 2$ m, $b = 4$ m)

$= 12$ kNm

Similarly B.M. at F, considering beam BC as simply supported

$= \frac{W_2 . a . b}{L_2} = \frac{8 \times 2 \times 3}{5}$ $\quad (\because$ Here $a = 2$, $b = 3$ and $L_2 = 5$)

$= 9.6$ kNm

The B.M. at the centre of a simply supported beam CD, carrying u.d.l.

$= \frac{w \times L_3^2}{8} = \frac{3 \times 4^2}{8} = 6$ kNm.

Now the B.M. diagram due to vertical loads taking each span as simply supported can be drawn as shown in Fig. 15.20 (*b*).

(*ii*) *B.M. diagram due to support moments*

Let M_A, M_B, M_C and M_D are the supports moments at A, B, C and D respectively. But the end supports of a simply supported beam are not subjected to any bending moment. Hence the support moments at A and D will be zero.

$\therefore$ $M_A = 0$ and $M_D = 0$

To find the support moments at B and C, Clapeyron's equation of three moments in applied for ABC and for BCD.

(*a*) For spans AB and BC from equation of three moments, we have

$$M_A . L_1 + 2M_B (L_1 + L_2) + M_C . L_2 = \frac{6a_1\bar{x}_1}{L_1} + \frac{6a_2\bar{x}_2}{L_2}$$

or $$0 + 2M_B (6 + 5) + M_C \times 5 = \frac{6a_1\bar{x}_1}{6} + \frac{6a_2\bar{x}_2}{5}$$

or $$22M_B + 5M_C = a_1\bar{x}_1 + \frac{6}{5} a_2\bar{x}_2 \quad ...(i)$$

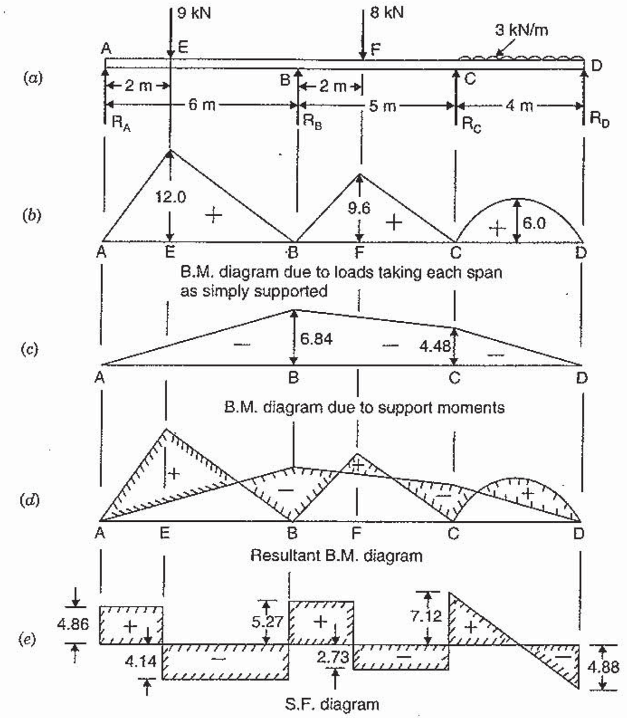


Fig. 15.20

Now $a_1\bar{x}_1$ = Moment of area of B.M. diagram due to vertical load on AB when AB is considered as simply supported beam about point A.

$$= \frac{1}{2} \times 2 \times 12 \times \frac{2 \times 2}{3} + \frac{1}{2} \times 4 \times 12 \times \left(2 + \frac{1}{3} \times 4\right)$$

$= 16 + 80 = 96$

$a_2\bar{x}_2$ = Moment of area of B.M. diagram due to vertical load on BC when BC is considered as simply supported beam about point C

$$= \frac{1}{2} \times 3 \times 9.6 \times \frac{2}{3} \times 3 + \frac{1}{2} \times 2 \times 9.6 \times \left(3 + \frac{1}{3} \times 2\right)$$

$= 28.8 + 35.2 = 64.0$

Substituting these values in equation (*i*), we get

$$22M_B + 5M_C = 96 + \frac{6}{5} \times 64$$

$$= 172.8 \quad \ldots(ii)$$

(*b*) For spans *BC* and *CD* from equation of three moments, we have

$$M_B . L_2 + 2M_C(L_2 + L_3) + M_D . L_3 = \frac{6a_2\bar{x}_2}{L_2} + \frac{6a_3\bar{x}_3}{L_3}$$

or $$M_B \times 5 + 2M_C(5 + 4) + 0 = \frac{6a_2\bar{x}_2}{5} + \frac{6a_3\bar{x}_3}{4} \quad (\because M_D = 0)$$

or $$5M_B + 18M_C = \frac{6}{5}a_2\bar{x}_2 + \frac{6}{5}a_3\bar{x}_3 \quad \ldots(iii)$$

where $a_2\bar{x}_2$ = Moment of area of B.M. diagram due to vertical load on *BC* when *BC* is considered as simply supported beam, about point *B*

$$= \frac{1}{2} \times 2 \times 9.6 \times \frac{2}{3} \times 2 + \frac{1}{2} \times 3 \times 9.6 \times \left(2 + \frac{1}{3} \times 3\right)$$

$$= 12.8 + 43.2 = 56.0$$

and $a_3\bar{x}_3$ = Moment of area of B.M. diagram due to u.d.l. on *CD*, when *CD* is considered as simply supported beam, about point *D*

$$= \left(\frac{2}{3} \times \text{Base} \times \text{Altitude}\right) \times \frac{\text{Base}}{2}$$

$$= \frac{2}{3} \times 4 \times 6 \times \frac{4}{2} = 32$$

Substituting these values in equation (*iii*), we get

$$5M_B + 18M_C = \frac{6}{5} \times 56 + \frac{6}{4} \times 32 = 115.2 \quad \ldots(iv)$$

Solving equations (*ii*) and (*iv*), we get

$$M_B = 6.84 \text{ kNm} \quad \text{and} \quad M_C = 4.48 \text{ kNm}.$$

Now the B.M. diagram due to supports moments is drawn as shown in Fig. 15.20 (*c*), in which

$$M_A = 0, M_B = 6.84, M_C = 4.48 \text{ and } M_D = 0.$$

The B.M. diagram due to supports moments will be negative. Resultant B.M. diagram is shown in Fig. 15.20 (*d*).

(*iii*) *Support Reactions*

Let R_A, R_B, R_C and R_D are the support reactions at *A*, *B*, *C* and *D* respectively,

For the span *AB*, taking moments about *B*, we get

$$M_B = R_A \times 6 - 9 \times 4$$

or $$-6.84 = 6R_B - 36 \quad (\because M_B = -6.84)$$

or $$R_B = \frac{36 - 6.84}{6} = \mathbf{4.86 \text{ kN.}} \quad \textbf{Ans.}$$

For the span *CD*, taking moments about *C*, we get

$$M_C = R_D \times 4 - 3 \times 4 \times \frac{4}{2}$$

or $$-4.48 = 4R_D - 24 \quad (\because M_C = -4.48)$$

$$\therefore \quad R_D = \frac{24 - 4.48}{4} = \mathbf{4.88 \text{ kN.}} \quad \textbf{Ans.}$$

Now taking moments about *C* for *ABC*, we get

$$M_C = R_A \times (6 + 5) - 9\,(5 + 4) + R_B \times 5 - 8 \times 3$$

or $$-4.48 = 4.86 \times 11 - 9 \times 9 + R_B \times 5 - 24 \quad (\because M_C = -4.48, R_A = 4.86)$$

$$\therefore \quad 5R_B = 81 + 24 - 4.86 \times 11 - 4.48 = 47.06$$

$$\therefore \quad R_B = \frac{47.06}{5} = \mathbf{9.41 \text{ kN.}} \quad \textbf{Ans.}$$

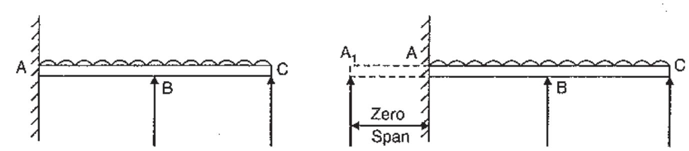
Now $$R_C = \text{Total load on } ABCD - (R_A + R_B + R_D)$$

$$= (9 + 8 + 4 \times 3) - (4.86 + 9.41 + 4.88)$$

$$= \mathbf{9.85 \text{ kN.}} \quad \textbf{Ans.}$$

Now complete the S.F. diagram as shown in Fig. 15.20 (*e*).

15.9.3. Clapeyron's Equation of Three Moments Applied to Continuous Beam with Fixed end Supports. We have seen in Art. 15.9.2 that fixing moments on the ends of a simply supported continuous beam are zero. But in case of a continuous beam fixed at its one or both ends, there will be fixing moments at the ends, which are fixed. To analyse the continuous beam which is fixed at the ends by the equation of three moments an imaginary support of zero span is introduced. The fixing moment at this imaginary support is always equal to zero.



(*a*) Continuous beam fixed at A (*b*) Continuous beam with zero span

Fig. 15.21

If the beam is fixed at the left end *A*, than an imaginary zero span is introduced to the left of *A* as shown in Fig. 15.21 (*b*). But if the beam is fixed at the right end, then an imaginary zero span is introduced to the right end support. After this Clapeyron's equation of three moments is applied.

Problem 15.12. *A continuous beam ABC of uniform section, with span AB and BC as 4 m each, is fixed at A and simply supported at B and C. The beam is carrying a uniformly distributed load of 6 kN/m run throughout its length. Find the support moments and the reactions. Also draw the bending moment and S.F. diagrams.*

Sol. Given :

Length *AB*, $L_1 = 4$ m

Length *BC*, $L_2 = 4$ m

U.d.l., $w = 6$ kN/m.

(*i*) *B.M. diagram due to u.d.l. taking each span as simply supported*

Consider beam *AB* as simply supported. The B.M. at the centre of the span *AB*

$$= \frac{w . L_1^2}{8} = \frac{6 \times 4^2}{8} = 12 \text{ kNm}$$

Similarly B.M. at the centre of span BC, considering beam BC as simply supported

$$= \frac{w \,.\, L^2}{8} = \frac{6 \times 4^2}{8} = 12 \text{ kNm}$$

The B.M. diagram due to u.d.l. taking each span as simply supported is drawn in Fig. 15.22 (*c*).

(*ii*) *B.M. diagram due to support moments*

As beam is fixed at A, therefore introduce an imaginary zero span AA_1 to the left of A as shown in Fig. 15.22 (*b*). The support moment at A_1 is zero.

Let M_0 = Support moment at A_1 and is zero

M_A = Support moment at A

M_B = Support moment at B

M_C = Support moment at C.

The extreme end C is simply supported hence $M_C = 0$. To find M_A and M_B theorem of three moments is used.

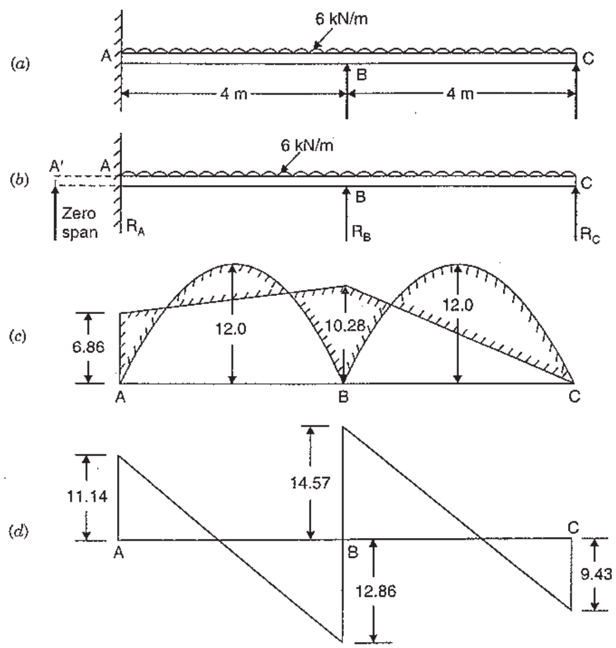


Fig. 15.22

Applying the theorem of three moments for the spans A_1A and AB, we have

$$M_0 \times 0 + 2M_A(0 + L_1) + M_B L_1 = \frac{6a_0\bar{x}_0}{L_0} + \frac{6a_1\bar{x}_1}{L_1}$$

or $$0 + 2M_A(0 + 4) + M_B \times 4 = 0 + \frac{6a_1\bar{x}_1}{4}$$

or $$8M_A + 4M_B = \frac{3}{2}\, a_1\bar{x}_1 \qquad \text{...}(i)$$

where $a_1\bar{x}_1$ = Moment of area of B.M. diagram due to u.d.l. on AB when AB is considered as simply supported beam about point B

$$= \left(\frac{2}{3} \times \text{Base} \times \text{Altitude}\right) \times \frac{L_1}{2}$$

$$= \frac{2}{3} \times 4 \times 12 \times \frac{4}{2} = 64.$$

Substituting this value in equation (*i*), we get

$$8M_A + 4M_B = \frac{3}{2} \times 64 = 96$$

or $$2M_A + M_B = 24 \qquad \text{...}(ii)$$

Now applying the theorem of three moments for the spans AB and BC, we get

$$M_A \,.\, L_1 + 2M_B\,(L_1 + L_2) + M_C \,.\, L_2 = \frac{6a_1\bar{x}_1}{L_1} + \frac{6a_2\bar{x}_2}{L_2}$$

or $$M_A \times 4 + 2M_B(4 + 4) + 0 = \frac{6a_1\bar{x}_1}{4} + \frac{6a_2\bar{x}_2}{4} \qquad (\because\; M_C = 0)$$

or $$4M_A + 16M_B = \frac{3}{2}\, a_1\bar{x}_1 + \frac{3}{2}\, a_2\bar{x}_2 \qquad \text{...}(iii)$$

where $a_1\bar{x}_1$ = Moment of area of B.M. diagram due to u.d.l. on AB when AB is considered as simply supported beam about C

$$= \frac{2}{3} \times 4 \times 12 \times \frac{4}{2} = 64$$

$a_2\bar{x}_2$ = Moment of area of B.M. diagram due to u.d.l. on BC when BC is considered as simply supported beam about C

$$= \frac{2}{3} \times 4 \times 12 \times \frac{4}{2} = 64.$$

Substituting these values in equation (*iii*), we get

$$4M_A + 16M_B = \frac{3}{2} \times 64 + \frac{3}{2} \times 64 = 192$$

or $$M_A + 4M_B = 48$$

Multiplying the above equation by 2, we get

$$2M_A + 8M_B = 96 \qquad \text{...}(iv)$$

Subtracting equation (*ii*) from equation (*iv*), we get

$$7M_B = 96 - 24 = 72$$

or $$M_B = \frac{72}{7} = \textbf{10.28 kNm.} \quad \textbf{Ans.}$$

Substituting this value in equation (*ii*), we get

$$2M_A + 10.28 = 24$$

or $$M_A = \frac{24 \times 10.28}{2} = \textbf{6.86 kNm.} \quad \textbf{Ans.}$$

Now B.M. diagram due to support moments is drawn as shown in Fig. 10.22 (c) in which $M_B = 6.86$, $M_B = 10.28$, and $M_C = 0$. The B.M. due to supports moments will be negative. Resultant B.M. diagram is also shown in Fig. 15.22 (c).

(iii) Support Reactions

Let R_A, R_B and R_C are the support reactions at A, B and C respectively.

For the span BC, taking moments about B, we get

$$M_B = R_C \times 4 - 6 \times 4 \times \frac{4}{2}$$

or $$-10.28 = 4R_C - 48 \qquad (\because \; M_B \text{ is negative})$$

or $$R_C = \frac{48 - 10.28}{4} = \mathbf{9.43 \ kN.} \quad \textbf{Ans.}$$

For the span AB, taking moments about B, we get

$$M_B = M_A + R_A \times 4 - 6 \times 4 \times \frac{4}{2}$$

or $$-10.28 = -6.86 + 4R_A - 48 \qquad (\because \; M_B \text{ and } M_A \text{ are negative})$$

or $$R_A = \frac{48 + 6.86 - 10.28}{4} = \mathbf{11.14 \ kN.} \quad \textbf{Ans.}$$

and $$R_B = \text{Total load} - (R_A + R_C)$$
$$= 6 \times 8 - (11.14 + 9.43) = \mathbf{27.43 \ kN.} \quad \textbf{Ans.}$$

Now complete the S.F. diagram as shown in Fig. 15.22 (*d*).

Problem 15.13. *A continuous beam ABC of uniform section, with span AB and BC as 6 m each, is fixed at A and C and supported at B as shown in Fig. 15.23 (a). Find the support moments and the reactions. Draw the S.F. and B.M. diagrams of the beam.*

Sol. Given :

Length AB, $L_1 = 6$ m

Length BC, $L_2 = 6$ m

U.d.l. in AB, $w = 2$ kN/m

Point load in BC, $W = 12$ kN.

(i) B.M. diagram due to vertical loads taking each span as simply supported

Consider beam AB as simply supported. The B.M. at the centre of AB

$$= \frac{w L_1^2}{8} = \frac{2 \times 6^2}{8} = 9 \text{ kNm.}$$

Consider beam BC as simply supported. The B.M. at the centre of BC

$$= \frac{W \times L_2}{4} = \frac{12 \times 6}{4} = 18 \text{ kNm}$$

The B.M. diagram due to vertical loads is drawn as shown in Fig. 15.23 (*c*).

(ii) B.M. diagram due to support moments

As beam is fixed at A and C, therefore introduce an imaginary zero span AA_1 and CC_1 to the left of A and to the right of C respectively as shown in Fig. 15.23 (*b*). The support moments at A_1 and C_1 are zero.

Let M_0 = Support moment at A_1 and C_1 and it is zero

M_A = Fixing moment at A

M_B = Support moment at B

M_C = Fixing moment at C.

To find M_A, M_B and M_C, theorem of three moments is used.

(*a*) Applying the theorem of three moments for the spans A_1A and AB, we get

$$M_0 \times 0 + 2M_A (0 + L_1) + M_B . L_1 = \frac{6a_0\bar{x}_0}{L_0} + \frac{6a_1\bar{x}_1}{L_1}$$

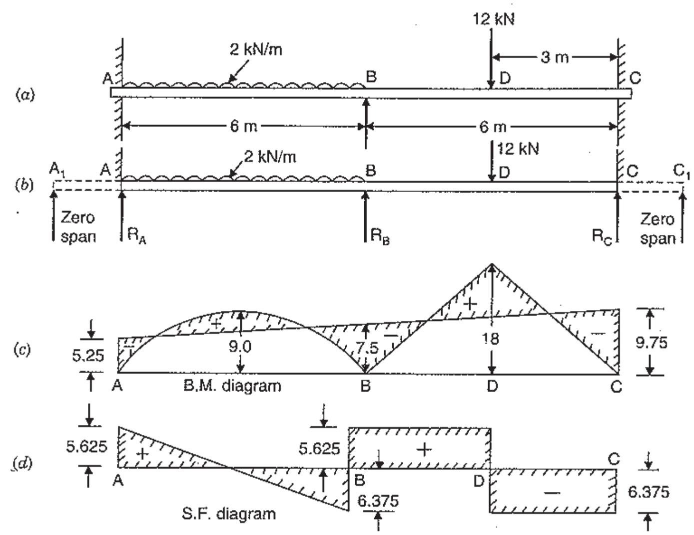


Fig. 15.23

or $$0 + 2M_A(0 + 6) + M_B \times 6 = 0 + \frac{6a_1\bar{x}_1}{6}$$

or $$12M_A + 6M_B = a_1\bar{x}_1 \qquad ...(i)$$

where $a_1\bar{x}_1$ = Moment of area of B.M. diagram due to u.d.l. on AB when it is considered as simply supported beam about B.

$$= \frac{2}{3} \times \text{Base} \times \text{Altitude} \times \frac{L_1}{2}$$

$$= \frac{2}{3} \times 6 \times 9 \times \frac{6}{2} = 108.$$

Substituting this value in equation (*i*), we get

$$12M_A + 6M_B = 108$$

or $$2M_A + M_B = 18 \qquad ...(ii)$$

(*b*) Applying the theorem of three moments for the span AB and BC, we get

$$M_A . L_1 + 2M_B(L_1 + L_2) + M_C.L_2 = \frac{6a_1\bar{x}_1}{L_1} + \frac{6a_2\bar{x}_2}{L_2}$$

or $$M_A \times 6 + 2M_B(6+6) + M_C \times 6 = \frac{6a_1\bar{x}_1}{L_1} + \frac{6a_2\bar{x}_2}{6}$$

or $$6M_A + 24M_B + 6M_C = a_1\bar{x}_1 + a_2\bar{x}_2 \quad ...(iii)$$

where $a_1\bar{x}_1 = \frac{2}{3} \times 6 \times 9 \times \frac{6}{2} = 108$

$a_2\bar{x}_2$ = Moment of area of B.M. diagram due to point load on BC when it is considered as simply supported beam about C

$= \frac{1}{2} \times 6 \times 18 \times 3 = 162$

Substituting these values in equation (*iii*), we get

$$6M_A + 24M_B + 6M_C = 108 + 162 = 270$$

or $$M_A + 4M_B + M_C = 45 \quad ...(iv)$$

(*c*) Now applying the theorem of three moments for the span BC and CC_1, we get

$$M_B.L_2 + 2M_C(L_2 + 0) + M_0 \times 0 = \frac{6a_2\bar{x}_2}{L_2} + \frac{6a_0\bar{x}_0}{L_0}$$

or $$M_B \times 6 + 2M_C(6+0) + 0 = \frac{6a_2\bar{x}_2}{6} + 0$$

or $$6M_B + 12M_C = a_2\bar{x}_2 \quad ...(v)$$

where $a_2\bar{x}_2$ = Moment of area of B.M. diagram due to point load on BC when it is considered as simply supported beam about B

$= \frac{1}{2} \times 6 \times 18 \times 3 = 162.$

Substituting this value in equation (*v*), we get

$$6M_B + 12M_C = 162 \quad ...(vi)$$

or $$M_B + 2M_C = 27$$

Solving equations (*ii*), (*iv*) and (*vi*), we get

$$M_A = 5.25 \text{ kNm}, \quad M_B = 7.5 \text{ kNm}$$

and $$M_C = 9.75 \text{ kNm}.$$

Now B.M. diagram due to support moments is drawn as shown in Fig. 15.23 (*c*). The B.M. due to support moments is negative.

(*iii*) *Support reactions*

Let R_A, R_B and R_C are the support reactions at A, B and C respectively.

For the span AB, taking moments about B, we get

$$M_B = R_A \times 6 - 6 \times 2 \times 3 + M_A$$

or $$-7.5 = R_A \times 6 - 36 - 5.25 \quad (\because M_B \text{ and } M_A \text{ are negative})$$

or $$R_A = \frac{36 + 5.25 - 7.5}{6} = \mathbf{5.625\ kN.} \quad \textbf{Ans.}$$

For the span BC, taking moments about B, we get

$$M_B = R_C \times 6 - 12 \times 3 + M_C$$

or $$-7.5 = R_C \times 6 - 36 - 9.75 \quad (\because M_B \text{ and } M_C \text{ are negative})$$

or $$R_C = \frac{36 + 9.75 - 7.5}{6} = \mathbf{6.375\ kN.} \quad \textbf{Ans.}$$

Now $$R_B = \text{Total load} - (R_A + R_C)$$
$$= (6 \times 2 + 12) - (5.625 + 6.375) = \mathbf{12\ kN.} \quad \textbf{Ans.}$$

The S.F. is shown in Fig. 15.23 (*d*).

HIGHLIGHTS

1. A beam whose both ends are fixed is known as fixed beam. And a beam which is supported on more than two supports is known as a continuous beam.
2. In case of a fixed beam :

   (*i*) $a = a'$ (*ii*) $a\bar{x} = a'\bar{x}'$ and $\bar{x} = \bar{x}'$

   Or

   (*i*) The area of B.M. diagram due to vertical loads is equal to the area of B.M. diagram due to end moments.

   (*ii*) Distance of C.G. of B.M. diagram due to vertical loads is equal to the distance of C.G. of B.M. diagram due to end moments from the same point.
3. The deflection at the centre of a fixed beam carrying a point load at the centre is given by

   $$y_c = \frac{WL^3}{192EI}$$

   where W = Point load,
   L = Length of beam.
4. The deflection at the centre of a fixed beam carrying a point load at the centre is one-fourth of the deflection of a simply supported beam.
5. The deflection of a fixed beam with an eccentric load, under the point load is given by,

   $$y_c = \frac{Wa^3b^3}{3EIL^3}.$$
6. (*a*) For a fixed beam carrying uniformly distributed load over the whole length :

   $$\text{End moments} = \frac{W \times L^2}{12}$$

   $$\text{Max. deflection} = \frac{w\,.\,L^4}{384EI}$$

   (*b*) The deflection at the centre of a fixed beam carrying uniformly distributed load over the whole span is one-fifth of the deflection of a simply supported beam.
7. The end moments of a fixed beam due to sinking of a support is given by

   $$M_A = M_B \frac{6EI\delta}{L^2}$$

   where δ = Sinking of one support with respect to the other. At the higher end this moment is –ve whereas at the lower end it is positive.
8. Clapeyron's theorem of three moments for a continuous beam ABC is given by,

   $$M_A.L_1 + 2M_B(L_1 + L_2) + M_C \times L_2 = \frac{6a_1\bar{x}_1}{L_1} + \frac{6a_2\bar{x}_2}{L_2}$$

   where a_1 = Area of B.M. diagram due to vertical loads on span AB
   a_2 = Area of B.M. diagram due to vertical loads on span BC
   $\bar{x}_1$ = Distance of C.G. of B.M. diagram due to vertical loads on AB from A
   $\bar{x}_2$ = Distance of C.G. of B.M. diagram due to vertical loads on BC from point C.

9. To apply the theorem of three moments to a fixed continuous beam, an imaginary support of zero span is introduced.

EXERCISE 15

(A) Theoretical Questions

1. What do you mean by a fixed beam and a continuous beam ?
2. Prove that for a fixed beam :
   (*i*) Area of B.M. diagram due to vertical loads is equal to the area of B.M. diagram due to end moments.
   (*ii*) Distance of C.G. of B.M. diagram due to vertical loads is equal to the distance of C.G. of B.M. diagram due to end moment from the same point.
3. Find an expression for the deflection for a fixed beam carrying a point load at the centre. Also obtain the value of maximum deflection.
4. Prove that the deflection at the centre of a fixed beam is one-fourth the deflection of a simply supported beam of the same length, when they carry a point load W at the centre.
5. Draw the S.F. and B.M. diagrams for a fixed beam, carrying an eccentric load.
6. Prove that the deflection at the centre of a fixed beam, carrying a uniformly distributed load is given by

$$y_c = \frac{wL^4}{384EI}$$

   Determine the position of points of contraflexures also.
7. Derive an expression for the fixing moments, when one of the supports of a fixed beam sinks down by δ from its original position.
8. What are advantages and disadvantages of a fixed beam over a simply supported beam ?
9. What is the Clapeyron's theorem of three moments ? Derive an expression for Clapeyron's theorem of three moments.
10. How will you apply Clapeyron's theorem of three moments to a
    (*i*) continuous beam with simply supported ends
    (*ii*) continuous beam with fixed end supports ?

(B) Numerical Problems

1. A fixed beam AB, 5 m long, carries a point load of 48 kN at its centre. The moment of inertia of the beam is 5×10^7 mm^4 and value of E for the beam material is 2×10^5 N/mm^2. Determine :
   (*i*) Fixed end moments at A and B, and
   (*ii*) Deflection under the load. [**Ans.** (*i*) $M_A = M_B = 30$ kNm, (*ii*) 3.125 mm]
2. A fixed beam of length 5 m carries a point load of 20 kN at a distance of 2 m from A. Determine the fixed end moments and deflection under the load, if the flexural rigidity of the beam is 1×10^4 kNm^2. [**Ans.** $M_A = 14.4$ kNm, $M_B = 9.6$ kNm, $y_C = 1.15$ mm]
3. A fixed beam of length 6 m carries point loads of 20 kN and 15 kN at distances 2 m and 4 m from the left end A. Find the fixed end moments and the reactions at the supports. Draw B.M. and S.F. diagrams. [**Ans.** $M_A = 24.44$ kNm, $M_B = 22.22$ kNm, $R_A = 18.70$ kN, $R_B = 16.30$ kN]
4. A fixed beam of length 3 m carries two point loads of 30 kN each at a distance of 1 m from both the ends. Determine the fixing moments and draw the B.M. diagram. [**Ans.** $M_A = M_B = 20$ kNm]

5. A fixed beam AB of length 6 m carries a uniformly distributed load of 3 kN/m over the left half of the span together with a point load of 4 kN at a distance of 4.5 m from the left end. Determine the fixing end moments and the support reactions. [**Ans.** $M_A = 7.3$ kNm, $M_B = 6.2$ kNm, $R_A = 7.93$ kN, $R_B = 5.07$ kN]
6. A fixed beam AB of length 6 m is having moment of intertia $I = 5 \times 10^6$ mm^4. The support B sinks down by 6 mm. If $E = 2 \times 10^5$ N/mm^2 find the fixing moments. [**Ans.** $M_A = M_B = 1000$ Nm]
7. A continuous beam ABC of length 10 m rests on three supports A, B and C at the same level in which span $AB = 6$ m and span $BC = 4$ m. In span AB , there is a point load of 3 kN at a distance of 2 m from the end A, whereas in the span BC, there is a uniformly distributed load of 1 kN/m run over the whole length. Determine the support moments and support reactions. Draw S.F. and B.M. diagrams also. [**Ans.** (*i*) $M_A = M_C = 0$, $M_B = 2.4$ kNm, (*ii*) $R_A = 1.6$ kN, $R_B = 4$ kN, $R_C = 1.4$ kN]
8. A continuous beam consists of three successive span of 8 m, 10 m and 6 m and carries loads of 6 kN/m, 4 kN/m and 8 kN/m respectively on the spans. Determine the bending moments and reactions at the supports. [**Ans.** (*i*) $M_A = M_D = 0$, $M_C = 32.2$ kNm, $M_B = 40.16$ kNm, (*ii*) $R_A = 18.98$ kN, $R_B = 49.82$ kN, $R_C = 48.57$ kN, $R_D = 18.63$ kN]
9. A continuous beam ABC consists of two consecutive spans AB and BC of length 8 m and 6 m respectively. The beam carries a uniformly distributed load of 1 kN/m throughout its length. The end A is fixed and the end C is simply supported. Find the support moments and the reactions. Also draw the S.F. and B.M. diagrams. [**Ans.** (*i*) $M_A = 5.75$ kNm, $M_B = 4.5$ kNm, $M_C = 0$, (*ii*) $R_A = 4.15$ kN, $R_B = 7.6$ kN, $R_C = 2.25$ kN]
10. Draw the S.F. and B.M. diagram of a continuous beam ABC of length 10 m which is fixed at A and is supported on B and C. The beam carries a uniformly distributed load of 2 kN/m length over the entire length. The spans AB and BC are equal to 5 m each. [**Ans.** (*i*) $M_A = 3.57$ kNm, $M_B = 5.357$ kNm, $M_C = 0$, (*ii*) $R_A = 5.357$ kN, $R_B = 8.571$ kN, $R_C = 6.071$ kN]