Theories of Failure

7.8 Theories of Elastic Failure

As we know a material is said to be failed if elastic limit is exceeded or permanent deformation takes place i.e., when the elastic limit is reached or when material starts yielding. In tensile test, yield stress (design stress) can be easily determined but if member is subjected to various complex stresses then it is very difficult to know the point of yielding or fracture. So, to improve the design of machine component, various theories of failure are developed considering physical behaviour of material.

Generally, there are two modes of failure:

1. Yielding or ductile failure
2. Brittle failure or fracture failure

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7.8.1 Maximum Principal Stress Theory (Rankine's Theory)

According to this theory, the material subjected to complex stresses will fail when principal stress induced in material reaches to yield stress

$$\sigma_1 = \sigma_y \quad \text{... for tension}$$

and failure can occur in compression when least principal stress σ_3 reaches the elastic limit stress (i.e., yield stress) in compression.

$$\sigma_3 = \sigma_y \quad \text{... for compression}$$

For no failure, maximum principal stress developed in strained material should be less than equal to yield stress in uniaxial loading

$$\sigma_1 \leq \sigma_y$$

For design stress, suitable factor of safety is introduced.

So design stress $$\sigma_{1d} \leq \frac{\sigma_y}{FOS}$$

- If σ_y (tension) = σ_y (compression) then this theory may be represented graphically as
- Failure will take place if any point having coordinate (σ_1, σ_2) falls outside the failure envelope.

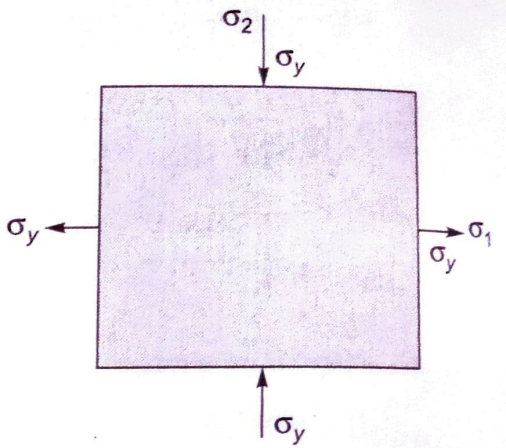


Fig. 7.28

Limitation:

- This theory neglect the effect of minor and intermediate principal stress.
- This theory is not suitable for ductile materials.
- Results of this theory are not suitable during hydrostatic loading.
- When material is subjected to pure shear, results for ductile materials are unsafe.

Hence, $\sigma_1 = +\tau_{max}$

$\sigma_2 = -\tau_{max}$

According to this theory,

$\sigma_1 \leq \sigma_y$

$\therefore \quad \tau_{max} \leq \sigma_y$

But experiment results show failure occurs much earlier when $\tau_{max} = 0.57\,\sigma_y$

So, for no failure in ductile material under pure shear, τ_{max} should be

$\tau_{max} \leq 0.57\,\sigma_y$

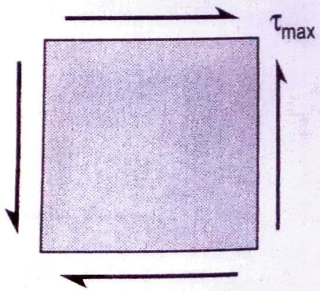
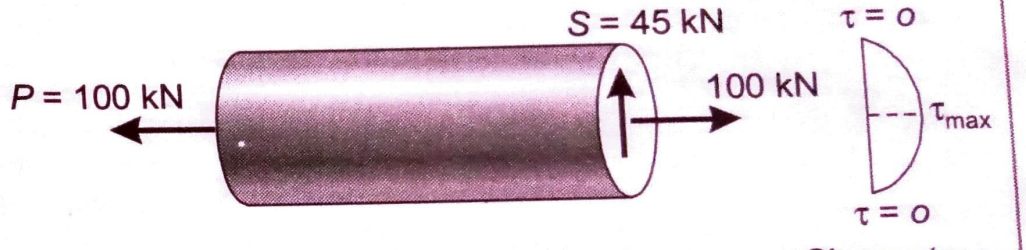


Fig. 7.29

Example 7.23 A prismatic circular bar is subjected to a pull of 100 kN and transverse shear of 45 kN. Using maximum principal stress theory, determine the dia. of bar. Take $\sigma_y = 300$ MPa and FOS = 3.

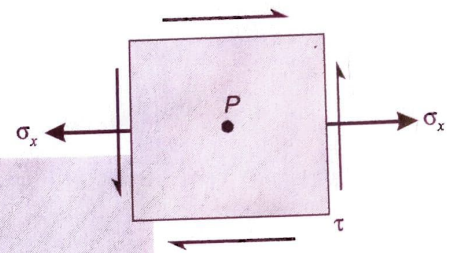


Solution:

Normal stress and shear stress on point P:

Normal stress, $$\sigma_x = \frac{100\times10^3}{A} = \frac{100\times10^3}{\frac{\pi\times d^2}{4}}$$

$$\sigma_x = \frac{127.32\times10^3}{d^2}$$



and $$\sigma_y = 0$$

Shear stress at the point *P*, $$\tau = \frac{4}{3}\tau_{avg} = \frac{4}{3}\times\frac{S}{A}$$

$$= \frac{4}{3}\times\frac{45\times10^3}{\frac{\pi\times d^2}{4}} = \frac{76.39\times10^3}{d^2}$$

Now point *P* can be considered as stress element subjected to σ_x and τ.

Major principal stress may be given as

$$\sigma_1 = \left(\frac{\sigma_x+\sigma_y}{2}\right) + \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau^2} = \left(\frac{127.32\times10^3+0}{2\times d^2}\right) + \sqrt{\left(\frac{127.32\times10^3-0}{2\times d^2}\right)^2+\left(\frac{76.39\times10^3}{d^2}\right)^2}$$

$$= \frac{63.66\times10^3}{d^2} + \sqrt{\left(\frac{63.66\times10^3}{d^2}\right)^2+\left(\frac{76.39\times10^3}{d^2}\right)^2} = \left(\frac{63.66}{d^2}+\frac{99.43}{d^2}\right)\times10^3 = \frac{163.09\times10^3}{d^2}$$

According to maximum principal stress theory

$$\sigma_1 \le \frac{\sigma_y}{FOS}$$

$$\frac{163.09\times10^3}{d^2} \le \frac{300}{3}$$

$$d^2 \ge \frac{163.09\times10^3}{100}$$

$$d^2 \ge 1630.90$$

$$\therefore \quad d = 40.38 \text{ mm}$$

7.8.2 Maximum Principal Strain Theory (St. Venant's Theory)

According to this theory, failure occurs when maximum strain (principal strain) in the complex stress system equals to the yield strain under uniaxial loading.

$$\epsilon_1 = \epsilon_y$$

Where,

$$\epsilon_1 = \frac{\sigma_1}{E} - \frac{\mu\sigma_2}{E} - \frac{\mu\sigma_3}{E}$$

and

$$\epsilon_y = \frac{\sigma_y}{E}$$

For no failure,

$$\epsilon_1 \le \epsilon_y$$

$$\epsilon_1 \le \frac{\sigma_y}{E}$$

For design:

$$\epsilon_1 \le \frac{\sigma_P}{E}$$

where,

σ_P = permissible stress

$$= \frac{\sigma_y}{FOS}$$

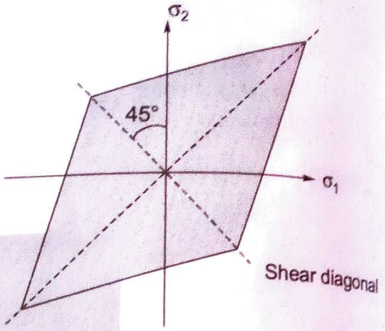


Fig. 7.30

- This theory can be applied for ductile and brittle materials both but results are not accurate for either case.

Limitations:

- In hydrostatic loading the results are not accurate.
- In case of pure shear, results are still unsafe for ductile materials but better than maximum principal stress theory

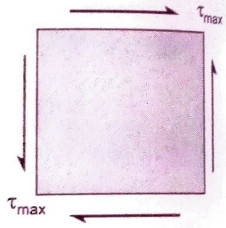


Fig. 7.31

Hence $\sigma_1 = \tau_{max}$

and $\sigma_2 = -\tau_{max}$

According to this theory

$$\epsilon \le \frac{\sigma_y}{E}$$

$$\frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E} \le \frac{\sigma_y}{E}$$

$$\sigma_1 - \mu\sigma_2 \le \sigma_y$$

$\therefore$

$$\tau_{max} + \mu\tau_{max} \le \sigma_y$$

$$\tau_{max}(1+\mu) \le \sigma_y$$

$$\tau_{max} \le \frac{\sigma_y}{1+\mu}$$

For ductile materials, $\mu = 0.3$

$$\tau_{max} \le 0.77\,\sigma_y$$

Which is still unsafe.

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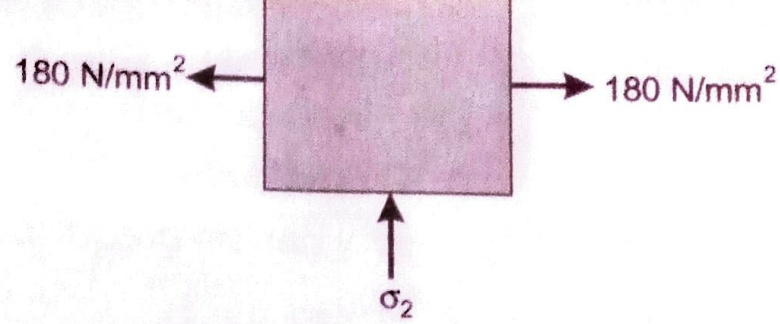
Example 7.24 In a two dimensional stress system, the two principal stresses are σ_1 of 180 N/mm^2 (tensile) and σ_2 (compressive). For the material, yield stress in simple tension and compression is 240 N/mm^2 and Poisson's ratio is 0.25. According to maximum normal strain theory, the value of σ_2 at which yielding will commence is

σ_2

Given

$\sigma_y = 240\ N/mm^2$

$\mu = 0.25$



According to maximum principal strain theory

$$\epsilon_1 \le \frac{\sigma_y}{E}$$

(in the direction of tensile stress)

$$\frac{\sigma_1}{E} + \mu\frac{\sigma_2}{E} \le \frac{\sigma_y}{E}$$

$$180 + 0.25\sigma_2 \le 240$$

$$0.25\sigma_2 \le 60$$

$$\sigma_2 \le 240\ N/mm^2 \quad \ldots(i)$$

and

$$\epsilon_2 \le \frac{\sigma_y}{E}$$

(in the direction of compressive stress)

$$-\frac{\sigma_2}{E} - \mu\frac{\sigma_1}{E} \le -\frac{\sigma_y}{E}$$

$$\sigma_2 + \mu\sigma_1 \le \sigma_y$$

$$\sigma_2 + 0.25 \times 180 \le 240$$

$$\sigma_2 \le 195\ N/mm^2 \quad \ldots(ii)$$

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For no yielding σ_2 will be lower of condition (i) and (ii)

$\therefore$ $\sigma_2 = 195\ N/mm^2$

7.8.3 Maximum Shear Stress Theory (Guest & Tresca's)

According to this theory, failure occurs when maximum shear stress in complex stress system equals to the max shear stress at yield point under uniaxial loading.

$$\tau_{max} = \frac{\sigma_y}{2}$$

For no failure, $$\tau_{max} \leq \frac{\sigma_y}{2}$$

and for design, $$\tau_{max} \leq \frac{(\sigma_y/FOS)}{2}$$

We know maximum shear stress in tri-axial loading will be max of $\frac{\sigma_1 - \sigma_2}{2}$, $\frac{\sigma_2 - \sigma_3}{2}$ and $\frac{\sigma_3 - \sigma_1}{2}$

and for 2D: $\frac{\sigma_1 - \sigma_2}{2}$

Hence for no failure $$\frac{\sigma_1 - \sigma_2}{2} \leq \frac{\sigma_y}{2} \quad \text{...[2D]}$$

$$(\sigma_1 - \sigma_2) \leq \sigma_y$$

- This theory is suitable for ductile materials and for pure shear case, it is oversafe.
- This theory can be graphically represented as figure 7.32

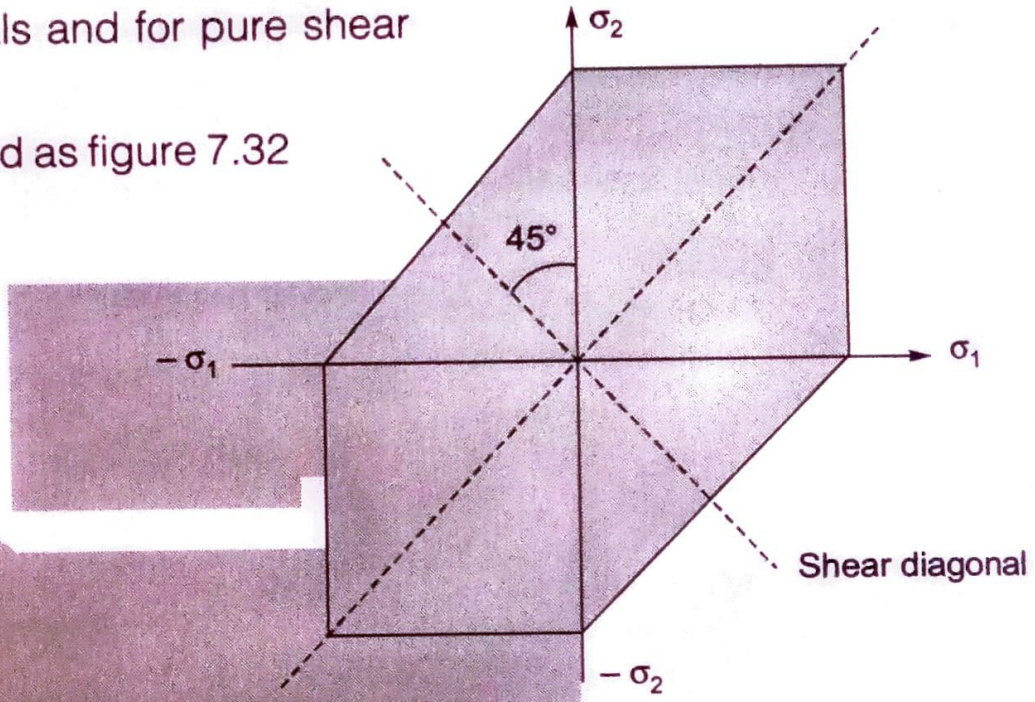

Fig. 7.32

Limitations:

- This theory is not applicable to materials subjected to hydrostatic loading. It says shear stress will be zero and material will never fail which is physically not possible.
- This theory cannot be applied for brittle materials as they have different yield stress in tension and compression.

Example 7.25 A certain steel has proportionality limit of 3000 kg/cm^2 in simple tension. It is subjected to principal stresses of 1200 kg/cm^2 (tensile), 600 kg/cm^2 (tensile) and 300 kg/cm^2 (compressive). The factor of safety according to maximum shear stress theory is

(a) 1.50 (b) 1.75 (c) 1.80 (d) 2.00

Ans. (d)

$\sigma_y = 3000\ \text{kg/cm}^2$

$\sigma_1 = 1200\ \text{kg/cm}^2$ (tensile)

$\sigma_2 = 600\ \text{kg/cm}^2$ (tensile)

$\sigma_3 = -300\ \text{kg/cm}^2$ (compressive)

$$\tau_{max} = \text{Max of } \left[\left|\frac{\sigma_1-\sigma_2}{2}\right|, \left|\frac{\sigma_2-\sigma_3}{2}\right|, \left|\frac{\sigma_3-\sigma_1}{2}\right|\right]$$

$$= \left[\left|\frac{1200-600}{2}\right|, \left|\frac{600+300}{2}\right|, \left|\frac{-300-1200}{2}\right|\right] = [300,\ 450,\ 750]$$

$\therefore \quad \tau_{max} = 750\ \text{N/mm}^2$

According to maximum shear stress theory

$$\tau_{max} \le \frac{(\sigma_y/\text{FOS})}{2}$$

$$750 \leq \frac{(300/FOS)}{2}$$

$$FOS \leq \frac{1500}{750}$$

$$\therefore \quad FOS = 2$$

7.8.4 Maximum Strain Energy Theory (Haigh and Beltrami)

According to this theory, failure occurs when maximum strain energy in complex stress system equals to the strain energy developed at yield stress in uniaxial loading

$$u = \frac{\sigma_y^2}{2E}$$

For no failure: $u \le \frac{\sigma_y^2}{2E}$

For design: $u \le \frac{(\sigma_y/FOS)^2}{2E}$

where, $u = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right]$ ...[3D]

or $u = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 - 2\mu\sigma_1\sigma_2\right]$ ...[2D]

Hence for no failure

$$\frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right] \le \frac{\sigma_y^2}{2E}$$

- It is suitable for ductile material
- This theory can be graphically represented as Figure 7.33

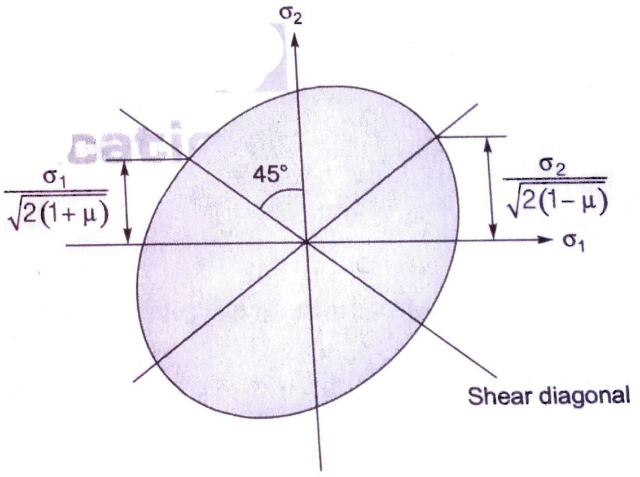

Fig. 7.33

Limitations:

- This theory can not be applied for brittle materials for which elastic limit stress in tension and compression are quite different.
- In case of pure shear, results are still unsafe for ductile materials

$$\sigma_1 = +\tau_{max}$$
$$\sigma_2 = -\tau_{max}$$

According to this theory,

$$\frac{1}{2E}\left(\sigma_1^2 + \sigma_2^2 - 2\mu\sigma_1\sigma_2\right) \le \frac{\sigma_y^2}{2E}$$

$$\tau_{max}^2 + \tau_{max}^2 + 2\mu\,\tau_{max}^2 \le \sigma_y^2$$

Example 7.26 The principal stress for machine component boiler are σ, 0.5σ and 0. If yield stress is found 240 MPa and $\mu = 0.3$. Find the suitable value of σ according to

1. Maximum principal strain theory 2. Maximum shear stress theory
3. Maximum strain energy theory

Solution:

Given, $\sigma_y = 240$ MPa, $\mu = 0.3$

$\sigma_1 = \sigma$

$\sigma_2 = 0.5\sigma$

and $\sigma_3 = 0$

1. Maximum principal strain theory

$$\epsilon_1 \le \frac{\sigma_y}{E}$$

$$\frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E} - \mu\frac{\sigma_3}{E} \le \frac{\sigma_y}{E}$$

$$\sigma - \mu \times \frac{\sigma}{2} - 0 \le 240$$

$$\sigma - \frac{0.3\sigma}{2} \le 240$$

$$0.85\sigma \le 240$$

$$\sigma = 282.35 \text{ N/mm}^2$$

2. Maximum shear stress theory

$$\tau_{max} \le \frac{\sigma_y}{2}$$

$$\left(\frac{\sigma_1 - \sigma_3}{2}\right) \le \frac{\sigma_y}{2} \quad \left\{\because (\sigma_1 - \sigma_3) > (\sigma_1 - \sigma_2) > (\sigma_2 - \sigma_3)\right\}$$

$$\sigma \le \sigma_y$$

$\Rightarrow$ $\sigma \le 240$

$\therefore$ $\sigma = 240 \text{ N/mm}^2$

3. Maximum strain energy theory

$$\frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right] \le \frac{\sigma_y^2}{2E}$$

$$\sigma^2 + \left(\frac{\sigma}{2}\right)^2 - 2 \times 0.3 \times \sigma \times \frac{\sigma}{2} \le (240)^2$$

$$\sigma^2 + \frac{\sigma^2}{4} - 0.3\sigma^2 \le 240^2$$

$$0.95\sigma^2 \le 240^2$$

$$\sigma = \sqrt{\frac{240^2}{0.95}} = 246.23 \text{ N/mm}^2$$

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Example 7.27 A bolt is under an axial thrust of 9 kN and transverse shear force of 5 kN. Calculate the diameter of bolt according to the theory of maximum strain energy. Take $\sigma_y = 250\ N/mm^2$, $\mu = 0.3$ and factor safety = 2.8. Assume uniform distribution of normal and shear stress on cross-section.

Solution:

Let d is the dia of bolt

Axial thrust, $P = 9 \times 10^3\ N$

Transverse shear force, $S = 5 \times 10^3\ N$

Axial stress, $$\sigma_x = -\frac{9\times10^3}{\left(\frac{\pi d^2}{4}\right)} = -\frac{36\times10^3}{\pi d^2}\ N/mm^2$$

Shear stress, $$\tau = \frac{5\times10^3}{\left(\frac{\pi d^2}{4}\right)} = \frac{20\times10^3}{\pi d^2}\ N/mm^2$$

Values of principal stresses may be given as

$$\sigma_1/\sigma_2 = \frac{\sigma_x}{2} \pm \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau^2} = \frac{\left(\frac{-36\times10^3}{\pi d^2}\right)}{2} \pm \sqrt{\frac{\left[\frac{-36\times10^3}{\pi d^2}\right]^2}{2} + \left(\frac{20\times10^3}{\pi d^2}\right)^2}$$

$$= \left[\frac{-18}{\pi d^2} \pm \sqrt{\left(\frac{18}{\pi d^2}\right)^2 + \left(\frac{20}{\pi d^2}\right)^2}\right]\times10^3 = \left[\frac{-18}{\pi d^2} \pm \frac{26.90}{\pi d^2}\right]\times10^3$$

$\therefore$ $$\sigma_1 = \frac{8.9\times10^3}{\pi d^2}$$

and $$\sigma_2 = -\frac{44.9\times10^3}{\pi d^2}$$

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According to maximum strain energy theory.

$$\sigma_1^2 + \sigma_2^2 - 2\mu\sigma_1\sigma_2 = \left(\frac{\sigma_y}{FOS}\right)^2$$

$$\left(\frac{8.9\times10^3}{\pi d^2}\right)^2 + \left(\frac{-44.9\times10^3}{\pi d^2}\right)^2 + \frac{2\times0.3\times8.9\times10^3\times44.9\times10^3}{\pi d^2\times\pi d^2} = \left(\frac{250}{2.8}\right)^2$$

$$\left(\frac{79.21\times10^6}{\pi^2 d^4}\right) + \left(\frac{2016.01\times10^6}{\pi^2 d^4}\right) + \left(\frac{239.76\times10^6}{\pi^2 d^4}\right) = 7971.93$$

$$\pi^2 d^4 = 292900.21$$

$$d^4 = 29676.99$$

$$d = 13.12\ mm$$

7.8.5 Maximum Shear Strain Energy Theory or Distortion Energy Theory (Mises-Henky Theory)

According to this theory, failure occurs when maximum shear strain energy stored in complex stress system equals to the shear strain energy stored at yield stress in uniaxial loading.

$$u_s = u_{ys}$$

Where u_s = shear stress energy stored due to complex stresses

and u_{ys} = shear strain energy stored due to yield stress

We know, total strain energy = volumetric strain energy + shear strain energy

$$u = u_v + u_s$$

Where

$$u = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right]$$

u_v = Vol. strain energy per unit volume

$$= \frac{1}{2} \times \sigma_{avg.} \times \text{volumetric strain}$$

$$= \frac{1}{2}\left(\frac{\sigma_1+\sigma_2+\sigma_3}{3}\right) \times \frac{\sigma_1+\sigma_2+\sigma_3}{E}(1-2\mu)$$

$\therefore$ $u_s = u - u_v$

Hence,

$$u_s = \frac{1}{3G}\left[\left(\frac{\sigma_1-\sigma_2}{2}\right)^2 + \left(\frac{\sigma_2-\sigma_3}{2}\right)^2 + \left(\frac{\sigma_3-\sigma_1}{2}\right)^2\right]$$

For 2D case:

$$u_s = \frac{1}{3G}\left[\left(\frac{\sigma_1-\sigma_2}{2}\right)^2 + \left(\frac{\sigma_2}{2}\right)^2 + \left(\frac{\sigma_1}{2}\right)^2\right]$$

$$u_s = \frac{1}{6G}\left(\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2\right)$$

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For no failure, $u_s \le \frac{\sigma_y^2}{6G}$

For design: $u_s \le \frac{(\sigma_y/FOS)^2}{6G}$

- This is most suitable for ductile metals
- For the case of pure shear, theoretical results are same as practical results.
- This theory can be graphically represented as Figure 7.34

Limitations:

- This theory cannot be applied for brittle materials
- It cannot be applied for materials under hydrostatic pressure.

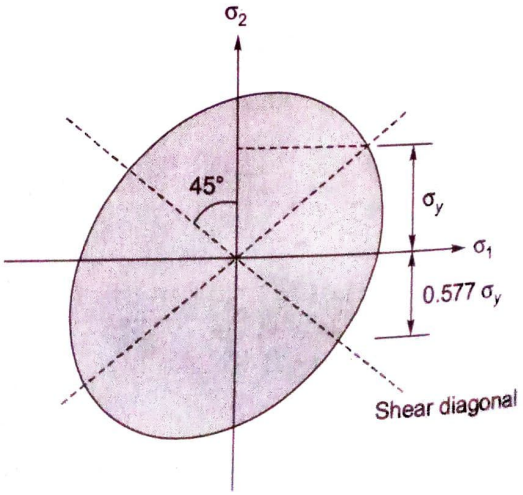

Fig. 7.34

7.8.6 Octahedral Shear Stress Theory

Octahedral planes are those plane which are equally incline to principal planes. Thus there are eight such planes which construct an octahedron. The shear stress along these plane is called **octahedral shear stress**.

The octahedral shear stress is given as

$$\tau_{oct} = \frac{2}{3}\left[\left(\frac{\sigma_1-\sigma_2}{2}\right)^2+\left(\frac{\sigma_2-\sigma_3}{2}\right)^2+\left(\frac{\sigma_3-\sigma_1}{2}\right)^2\right]^{1/2}$$

Special cases:

If $\sigma_3 = 0$ i.e., (2D case)

$$\tau_{oct} = \frac{\sqrt{2}}{3}\left[\sigma_1^2+\sigma_2^2-\sigma_1\sigma_2\right]^{1/2}$$

if $\sigma_2 = \sigma_3 = 0$ and $\sigma_1 = \sigma_y$

$$\tau_{oct} = \frac{\sqrt{2}}{3}\sigma_y$$

According to this theory, failure occurs when octahedral stress in complex stress system equals to the octahedral stress developed at yield stress in uniaxial loading.

$$\tau_{oct} = \frac{\sqrt{2}}{3}\sigma_y$$

For no failure: $\tau_{oct} \le \frac{\sqrt{2}}{3}\sigma_y$

For design: $\tau_{oct} \le \frac{\sqrt{2}}{3}\left(\frac{\sigma_y}{FOS}\right)$

For 2D:

$$\frac{\sqrt{2}}{3}\left[\sigma_1^2+\sigma_2^2-\sigma_1\sigma_2\right]^{1/2} \le \frac{\sqrt{2}}{3}\sigma_y$$

$$\left[\sigma_1^2+\sigma_2^2-\sigma_1\sigma_2\right]^{1/2} \le \sigma_y$$

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- In case of pure shear in ductile materials, theoretical results are same are practical result.

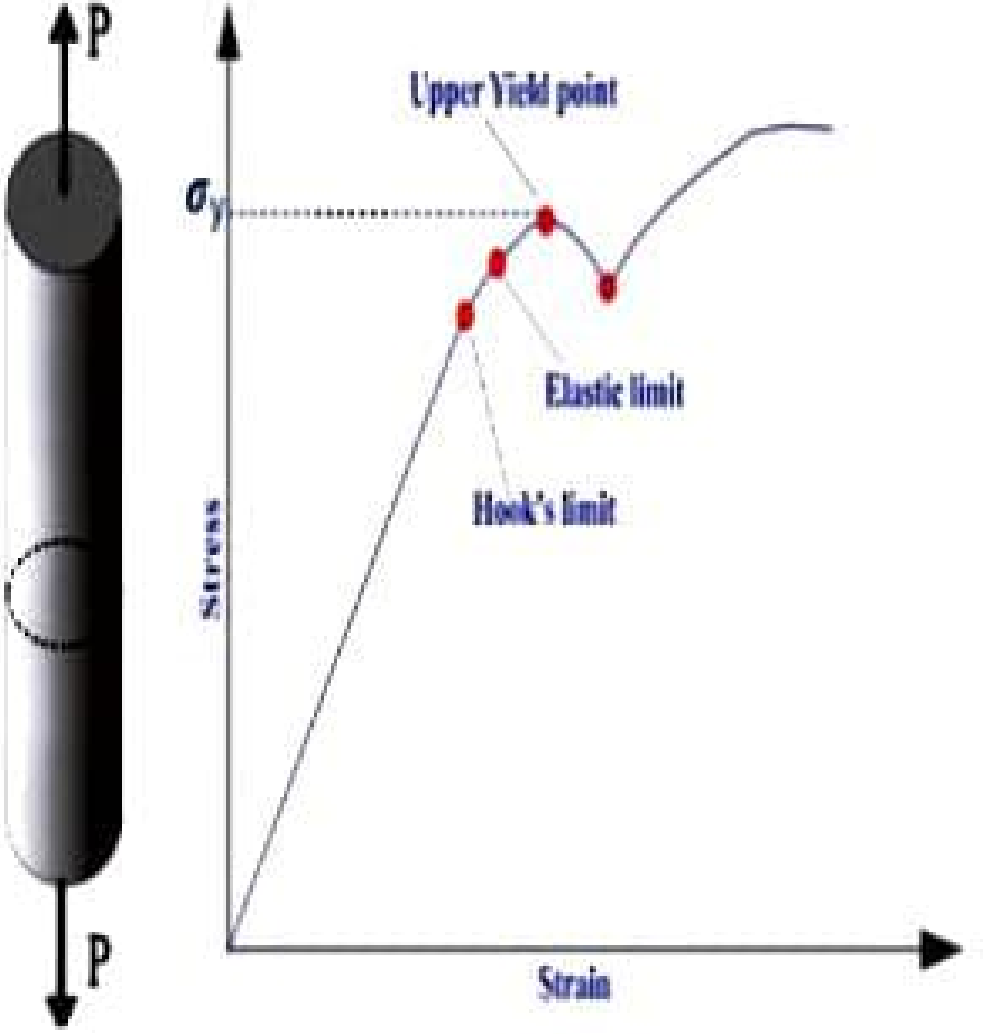
Introduction

Failing Factors

- Stress
  - Principal or Normal Stress
  - Shear Stress
- Strain → Principal Strain
- Energy
  - Maximum Strain Energy
  - Shear Strain Energy

Failure Criteria's

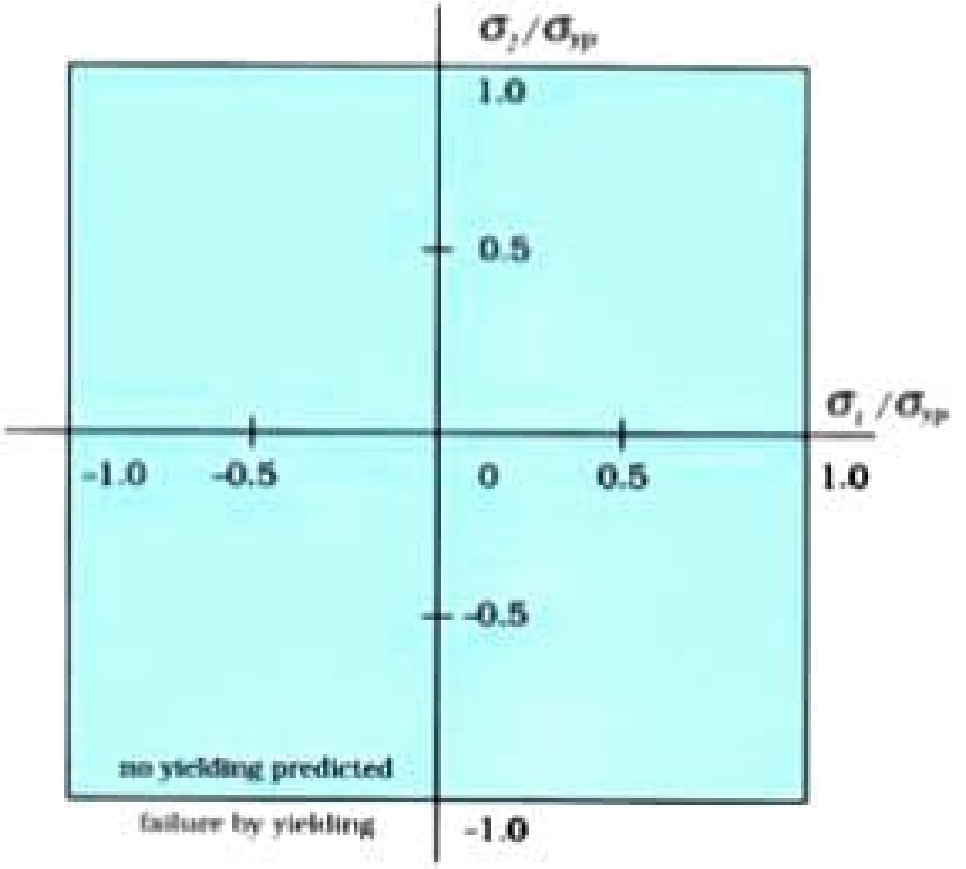
- Maximum Principal Stress
- Maximum Shear Stress
- Maximum Principal Strain
- Maximum Strain Energy Density
- Maximum Distortion Energy Density



Maximum Principal Stress Theory

When the maximum principal stress induced in a material under complex load condition exceeds the maximum normal strength in a simple tension test the material fails

Good for brittle materials



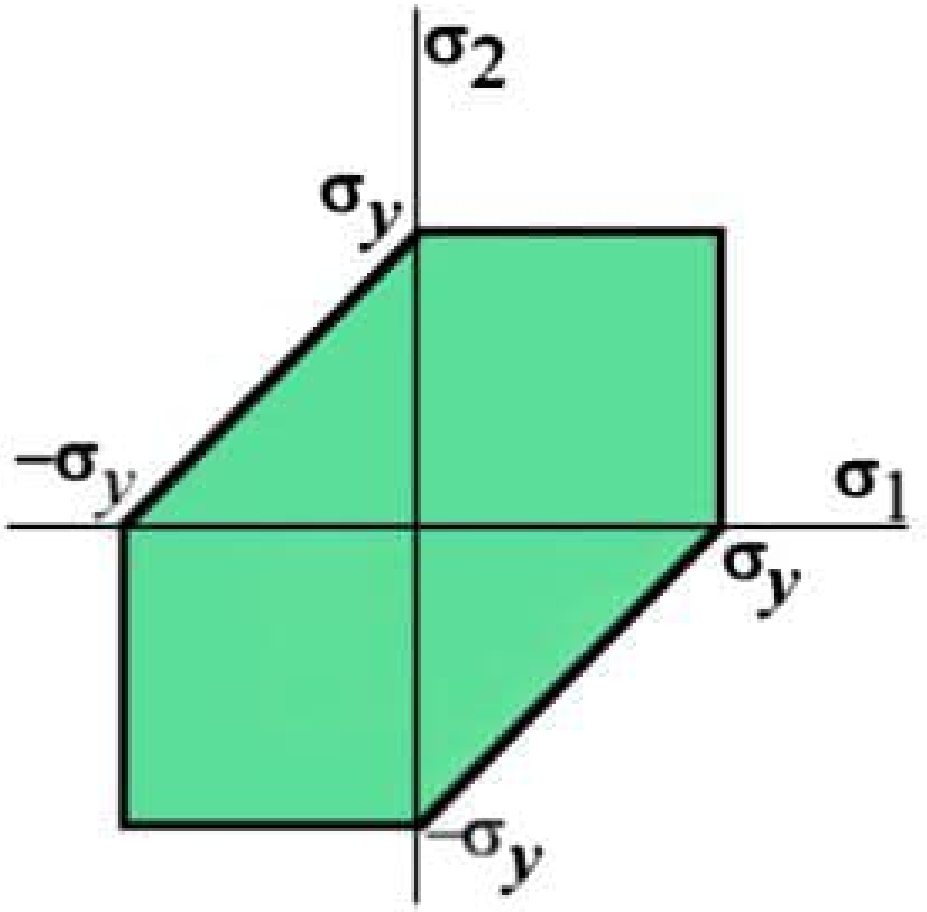
$$\sigma_1 \geq \sigma_{ult}$$

Maximum Shear Stress Theory

When the maximum shear strength in actual case exceeds maximum allowable shear stress in simple tension test the material case.

Good for ductile materials

$$\tau_{45} = \tau_{max\,simp} = \frac{\sigma_y}{2}$$

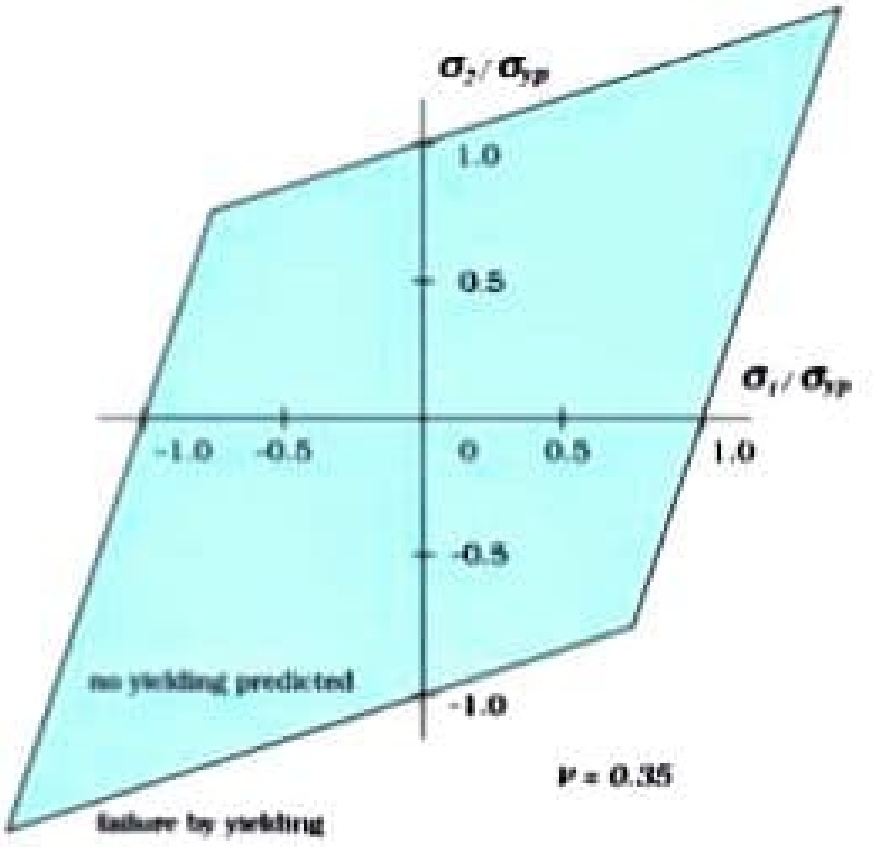


Maximum Principal Strain Theory

When the maximum normal strain in actual case is more than maximum normal strain occurred in simple tension test case the material fails

Not recommended

$$strain_{max} = \frac{\sigma_y}{E}$$

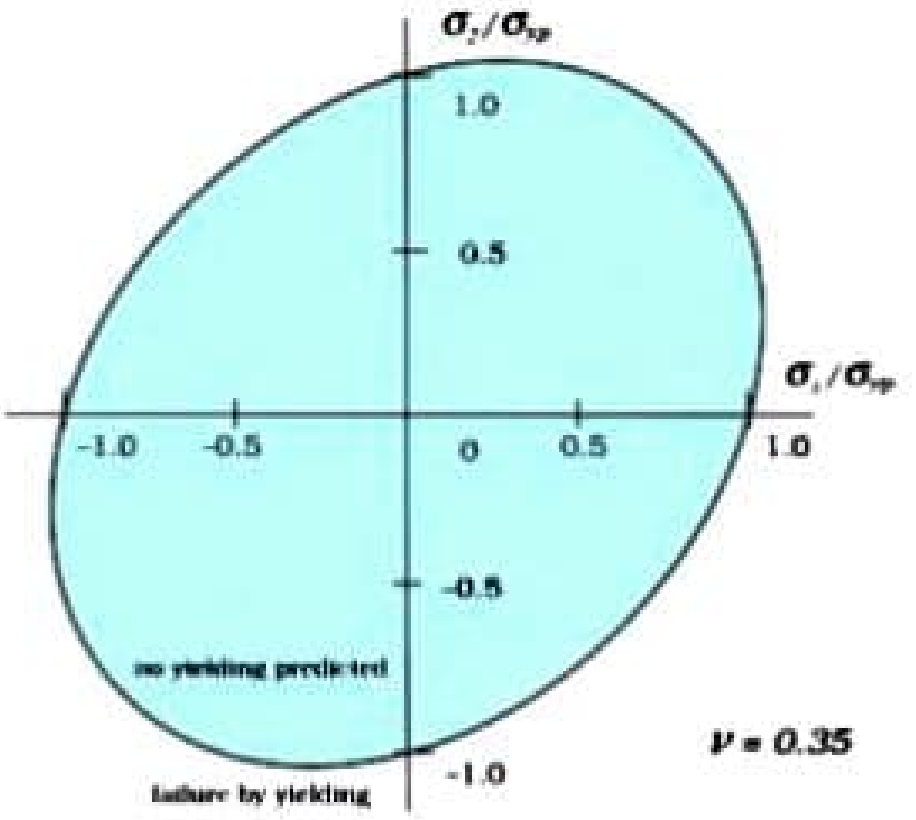


Maximum Strain Energy Density Theory

When the total strain energy in actual case exceeds the total strain energy in simple tension test at the time of failure, the material fails

Good for ductile material

$$T.S.E_{stmp} = \frac{{\sigma_y}^2}{2E}$$

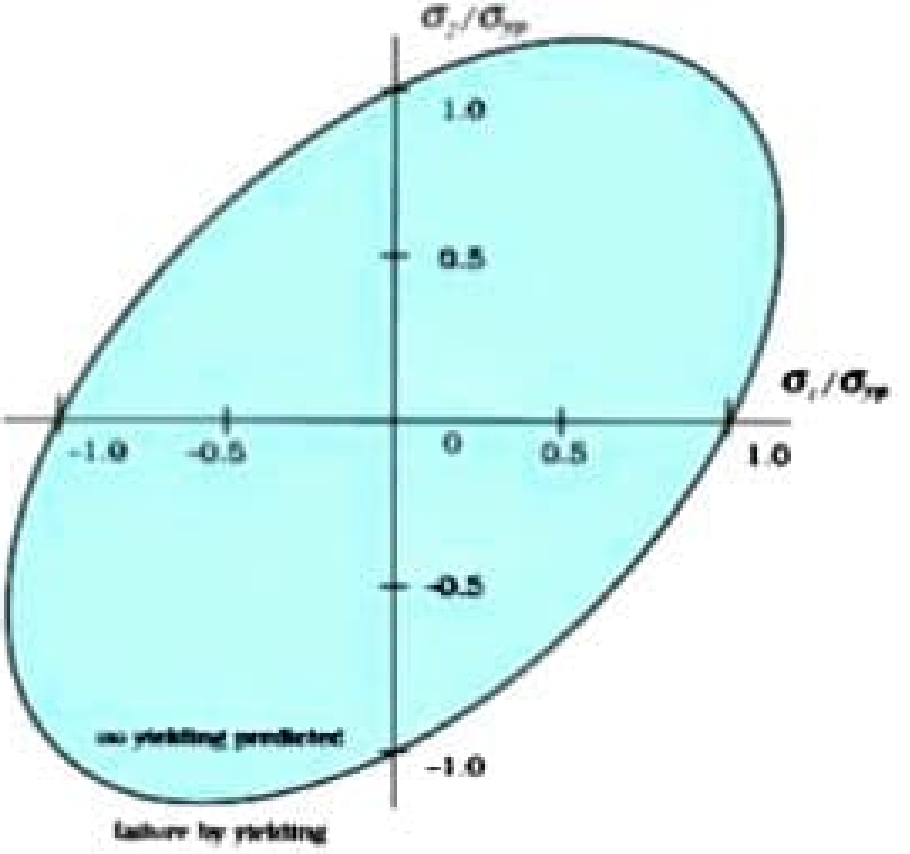


Maximum Distortion Energy Density Theory

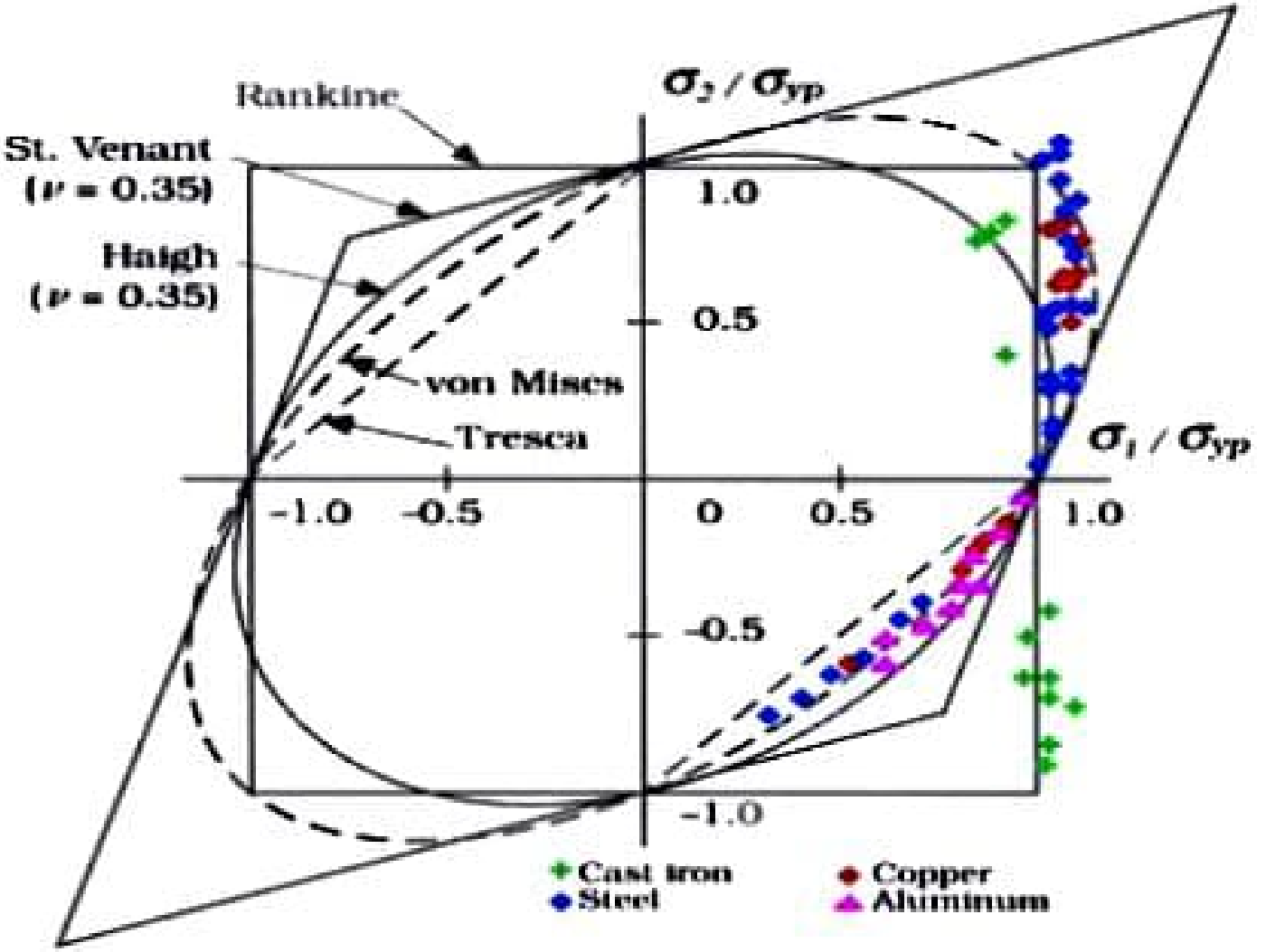
When the shear strain energy in the actual case exceeds shear strain energy in simple tension test at the time of failure the material fails

Highly recommended

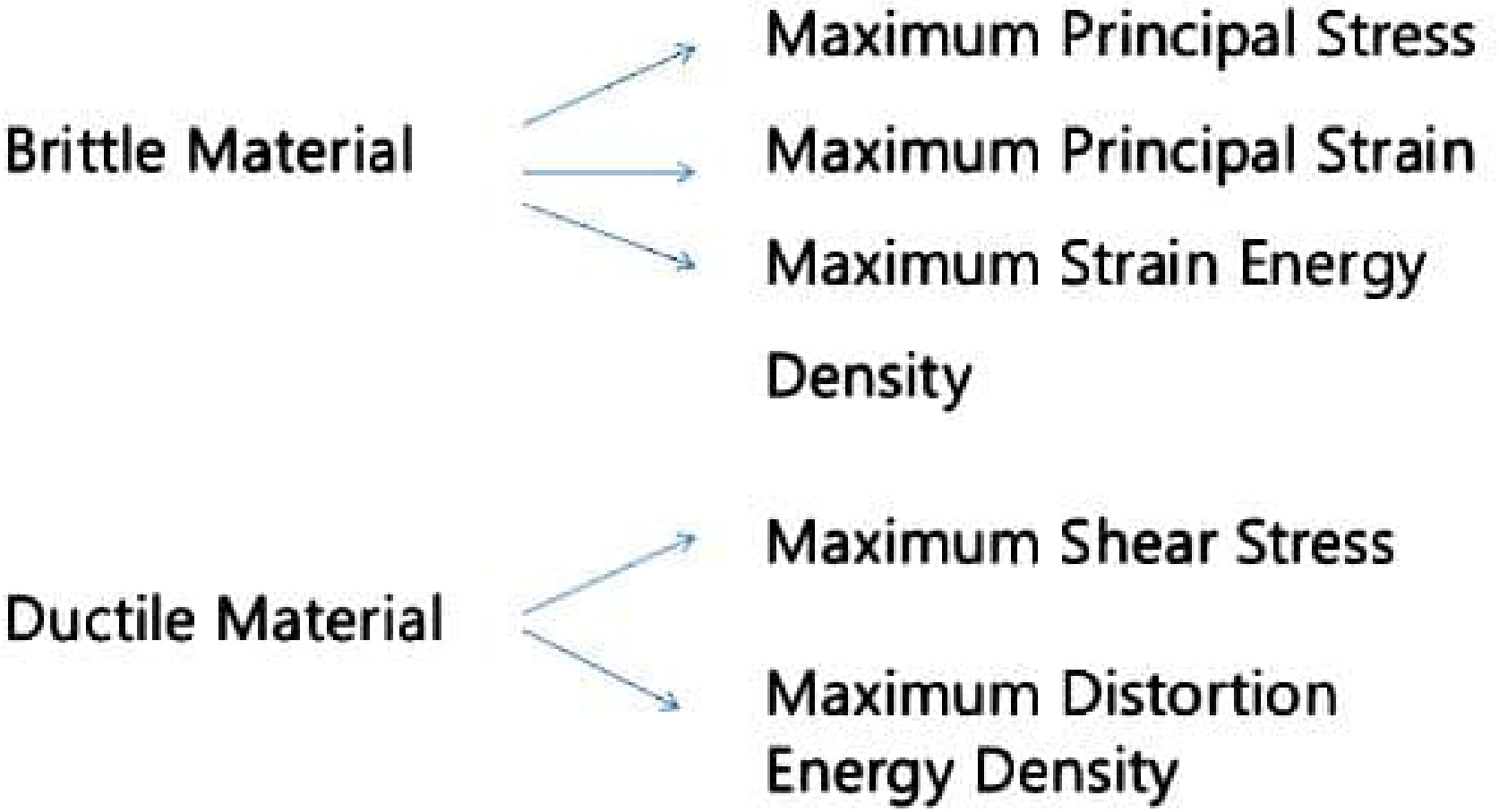
$$S.S.E_{simp} = \frac{{\sigma_y}^2}{6G}$$



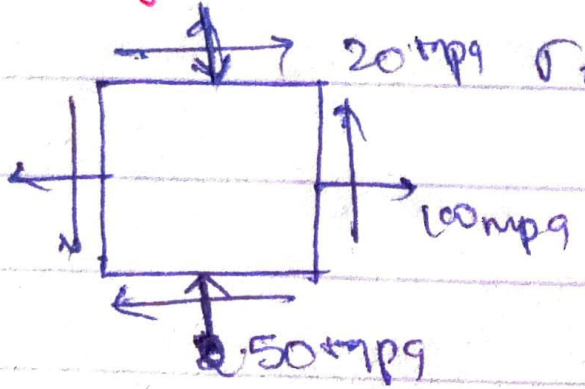
Comparison


Rankine
St. Venant
(ν = 0.35)
Haigh
(ν = 0.35)
von Mises
Tresca
σ2 / σyp
σ1 / σyp
1.0
0.5
-0.5
-1.0
-1.0
-0.5
0
0.5
1.0
Cast iron
Steel
Copper
Aluminum

Failure Criterion



Pb: → The state of stress @ a point in a material is shown in fig. [$\mu = 0.3$] [$E = 200$ GPa] determine the strength of the material to be used, in the design using Rankine & St. Venents theory?



$\sigma_x = 100 \quad \sigma_y = -50 \quad \tau_{xy} = 20$

$$\therefore \sigma_1 \& \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 25 \pm 77.62$$

$\therefore \sigma_1 = 102.62 \simeq 103$ mpa. $\quad \sigma_2 = -52.62$ mpa.

$\sigma_3 = -52.62$ mpa ≈ -53 mpa

$[\sigma_1 * f \Rightarrow 103 * f]$ — ① $\approx \boxed{f = 103 \text{ mpa}}$ (by Rankine's)

ii) St. Venants theory

$\sigma_1 - \mu\sigma_2 - \mu\sigma_3 * f$

$118.9 * f \approx \boxed{f = 118.9 \text{ mpa}}$

$\boxed{\text{Ans} = 118.9}$ take max strength for design.