Theory of Springs

12.1 Springs

The springs are the elastic members which deformed due to load and regain its original shape after removal of load. In spring, material is arranged in such a way that it can undergo a considerable change of shape, without getting permanently distorted. A spring is used to absorb energy in the form of resilience which may be restored when required. The quality of a spring is judged from the energy it can absorb and the natural frequency of oscillation. Natural frequency of oscillation should not be equal to the operating frequency of system otherwise resonance will take place.

12.2 Types of Springs

There are two types of springs:

(i) Bending springs

(ii) Torsional springs

12.2.1 Bending Springs

- Bending springs are the springs subjected to the bending moment only.
- The energy stored in bending spring is only due to bending.
- Examples: Laminated springs or leaf springs.

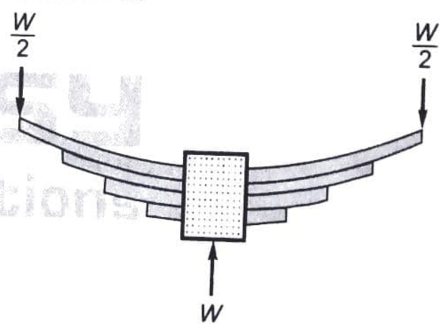


Fig. 12.1 Leaf spring

12.2.2 Torsional Spring

- Torsional springs are the springs subjected to the torsional moment only.
- The energy stored in torsional springs is only due to torsion.
- Examples: Open coil and closed coil helical springs.

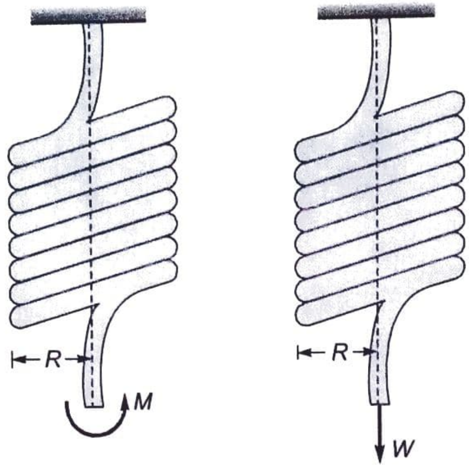


Fig. 12.2

12.2.3 Helical Spring

There are two types of helical springs:

(i) Closed coiled helical springs

(ii) Open coiled helical springs

(i) Closed coiled helical spring

In closed coiled helical springs, the wire or rod is wound closely in such a way that the pitch between two consecutive coil is very small. Closed coiled helical springs can be subjected to axial pull or axial twist.

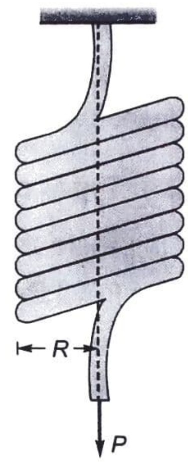


Fig. 12.3

Let spring is made of circular rod of length L, R is the mean radius of spring and d is diameter of rod from which spring is made. Also n is no. of turns in the spring.

Hence $$L = 2\pi Rn$$

Let spring is subjected to axial force P which acts through the centre of spring. Hence due to force P, there will be a torque $T = PR$ and shear force P on the section of metal rod. Let A is inner surface of spring and B is outer surface of spring.

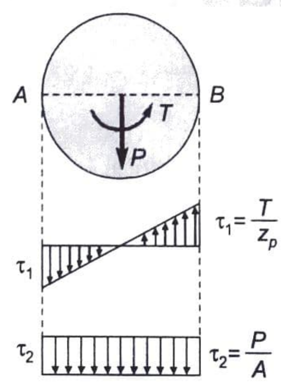


Fig. 12.4

At the section of spring shear stresses are produced due to torque and shear force. Let shear stress due to torque is τ_1 and average shear stress due to shear is τ_2

Resultant shear stress at,

$$\text{Inner surface} = \tau_1 + \tau_2$$

$$\therefore \quad \tau_{max} = \tau_1 + \tau_2 = \frac{T}{Z_P} + \frac{P}{A} = \frac{PR}{\left(\frac{\pi d^3}{16}\right)} + \frac{P}{\left(\frac{\pi d^2}{4}\right)} = \frac{4P}{\pi d^2}\left[\frac{4R}{d} + 1\right]$$

or

$$\tau_{max} \simeq \frac{T}{Z_P} = \frac{16PR}{\pi d^3} \qquad \left[\frac{4R}{d} \gg 1 \text{ and } \tau_1 \gg \tau_2\right]$$

$$\text{At outer surface} = \tau_1 - \tau_2$$

$$\therefore \quad \tau_{min} = \tau_1 - \tau_2$$

Since effect of shear force as compared to torque is negligible hence spring is called **torsional spring.**

Strain energy stored in closed coiled spring

The strain energy stored in closed coiled helical spring will be due to torsion only.

Hence,

$$U = \frac{T^2 L}{2GI_P} = \frac{(PR)^2 \times 2\pi Rn}{2G\frac{\pi d^4}{32}} = \frac{32P^2R^3n}{Gd^4} \Rightarrow \boxed{U = \frac{32P^2R^3n}{Gd^4}}$$

The axial deflection of spring under load *P*

According to Castigliano's theorem

$$\boxed{\Delta = \frac{\partial U}{\partial P} = \frac{64PR^3n}{Gd^4}}$$

Stiffness of closed coil spring

The load required to produce a unit deflection in a spring is called **stiffness of spring**. If k is coefficient of stiffness of spring

Then

$$k = \frac{P}{\Delta} = \frac{Gd^4}{64R^3n}$$

Example 12.1 A closed coil spring of mean diameter 80 mm is made of the high tensile rod of 10 mm diameter. Determine (i) shear stress (ii) axial deflection and (iii) stiffness of spring. When spring is subjected to axial load of 200 N. Take G = 82 GPa and number of turns is 18.

Solution:

Mean dia, $D = 80$ mm

$\therefore$ Mean radius, $R = 40$ mm

Diameter of rod $= 10$ mm

$G = 82 \times 10^3$ N/mm^2

(i) Shear stress,

$$\tau = \frac{16PR}{\pi d^3} = \frac{16 \times 200 \times 40}{\pi \times 10^3} = 40.74 \text{ N/mm}^2$$

(ii) Axial deflection,
$$\Delta = \frac{64PR^3n}{Gd^4} = \frac{64 \times 200 \times 40^3 \times 18}{82 \times 10^3 \times 10^4} = 17.98 \text{ mm}$$

(iii) Stiffness,
$$k = \frac{P}{\Delta} = \frac{200}{17.98} = 11.12 \text{ N/mm}$$

(ii) Open coiled helical spring subjected to axial load

In this case, load P will cause both twisting and bending of coils.

Deflection of spring due to axial load

$$\Delta = \frac{64PR^3 n \sec\alpha}{d^4}\left[\frac{\cos^2\alpha}{G} + \frac{2\sin^2\alpha}{E}\right]$$

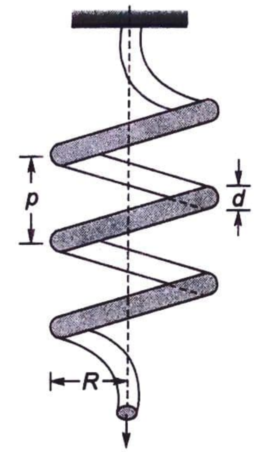


Fig. 12.5

where, d = diameter of spring wire or rod
R = mean radius of spring coil
p = pitch of spring coil
n = number of turns or coils
G = modulus of rigidity for spring material
α = angle of helix
E = Young's modulus

Bending Stress

$$\sigma = \frac{32PR\sin\alpha}{\pi d^3}$$

Shear stress

$$\tau = \frac{16PR\cos\alpha}{\pi d^3}$$

Example 12.2 **An open coiled helical spring made of 10 mm diameter steel rod of mean coil radius 50 mm and angle of helix 22° is subjected to an axial load P. Determine magnitude of load P if the maximum shear stress in wire due to torque is limited to 135 N/mm². Calculate number of turns in the spring if axial extension in the spring under the load P is 40 mm. Take G = 80 kN/mm² and E = 210 kN/mm².**

Solution:

$$d = 10 \text{ mm}$$
$$R = 50 \text{ mm}$$
$$\alpha = 22°$$
$$\tau_{max} = 135 \text{ N/mm}^2$$
$$\Delta = 40 \text{ mm}$$
$$P = \text{Load}$$

We know,
$$\tau_{max} = \frac{16PR\cos\alpha}{\pi d^3}$$

$$\Rightarrow \qquad 135 = \frac{16P \times 50 \times \cos 22°}{\pi \times 10^3}$$

$$\Rightarrow \qquad P = 571.78 \text{ N}$$

If axial extension in spring is 40 mm.

$$\Delta = \frac{64PR^3.n.\sec\alpha}{d^4}\left[\frac{\cos^2\alpha}{G} + \frac{2\sin^2\alpha}{E}\right]$$

$$\Rightarrow \qquad 40 = \frac{64 \times 571.78 \times 50^3 \sec 22°}{10^4}\left[\frac{\cos^2 22°}{80 \times 10^3} + \frac{2\sin^2 22°}{210 \times 10^3}\right]n$$

$$\Rightarrow \qquad n = 6.73 \simeq 7$$

12.3 Springs in Series and Parallel

1. **Spring in series:** If springs are in series, then force in each spring will be equal

   Then, total extension, $\Delta = \Delta_1 + \Delta_2$

$$\Rightarrow \qquad \frac{P}{k_1} + \frac{P}{k_2} = \frac{P}{k_{eq}}$$

$$\therefore \qquad \boxed{\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}}$$

   Where k_1, k_2 are individual stiffness of springs and k_{eq} is equivalent stiffness of combination.

Fig. 12.6

2. **Springs in parallel:** In parallel combination, force developed in each spring will be different but deflection in each spring will be equal. Let P is total force which is shared by spring 1 and 2

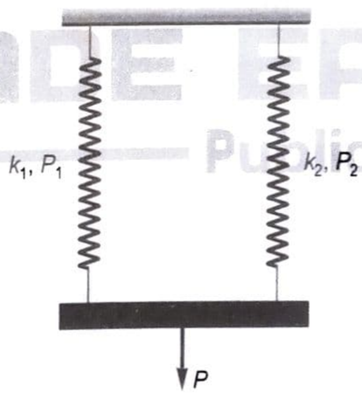


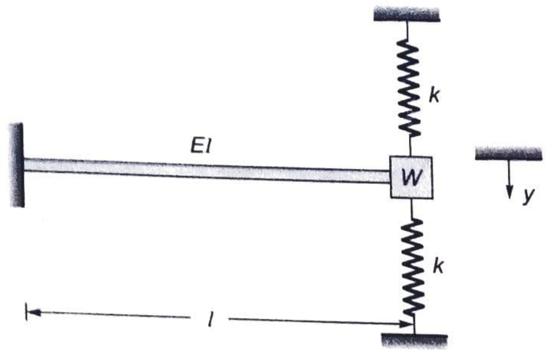
Fig. 12.7

$$P = P_1 + P_2$$

$$\Rightarrow \qquad k_{eq}\Delta = k_1\Delta + k_2\Delta$$

$$\therefore \qquad \boxed{k_{eq} = k_1 + k_2}$$

Example 12.3 For the beam shown below, the equivalent spring stiffness of the system is



(a) $\dfrac{3EI}{l^3} + 2k$

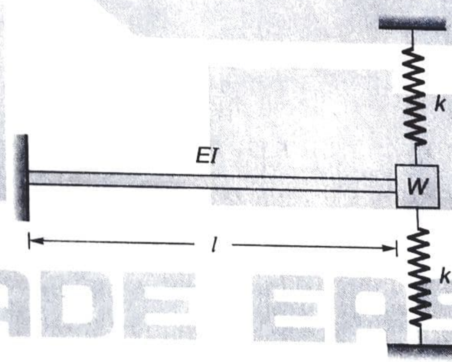
(b) $\dfrac{3EI}{l^3} + \dfrac{1}{2k}$

(c) $\dfrac{l^3}{3EI} + \dfrac{1}{2k}$

(d) $\dfrac{l^3}{3EI} + 2k$

Ans. (a)

Stiffness is defined as the force required to produce unit displacement.



Equivalent stiffness of system

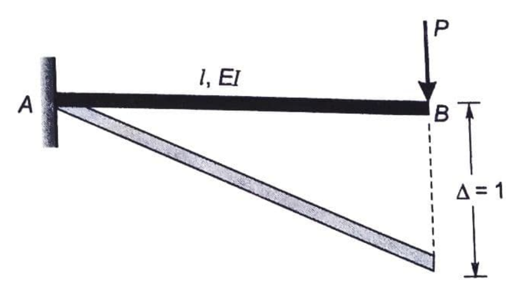
$$= k_{cantilever} + k_{spring}$$

$$k_{spring} = k + k \qquad [\because \text{Springs in parallel}]$$

$$= 2k$$

For stiffness of cantilever

$$\Delta_B = \frac{PL^3}{3EI}$$



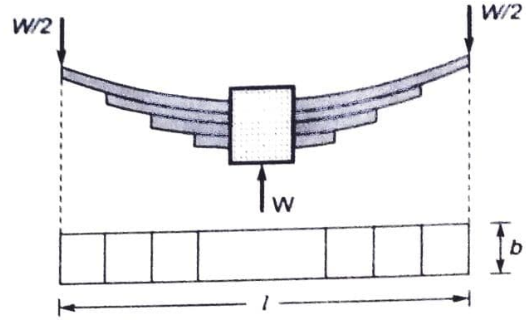
If $\qquad \Delta_B = 1,\ P = k = \dfrac{3EI}{L^3}$

Hence equivalent stiffness of system is given by

$$k_{eq} = \frac{3EI}{L^3} + 2k$$

Leaf Spring

Leaf springs are made from number of overlapping plates without any bond between them. All plates are initially bent to the same radius and free to slide over each other. Generally, leaf springs are loaded at ends and supported at centre



Where, l = Span of spring
t = Thickness of each plates
b = Width of plates
n = Number of plates
W = Load acting on the spring
E = Young's modulus

Fig. 12.8

Maximum bending moment at centre $= \frac{W}{2} \times \frac{l}{2} = \frac{Wl}{4}$

Bending moment resisted by each plate $= \frac{Wl}{4n}$

Maximum bending stress developed in each plate

$$\sigma_{max} = \frac{M}{Z} = \frac{WL/4n}{bt^2/6} = \frac{3}{2}\frac{Wl}{nbt^2}$$

$$\therefore \quad \boxed{\sigma_{max} = \frac{3\ Wl}{2\ nbt^2}}$$

Since each plate will bend about its own neutral axis. Hence each plate will be in tension and compression.

Maximum Deflection: The maximum deflection will occur at the ends (i.e., under the point of loading)

From geometry,
$$\delta = \frac{L^2}{8R} \quad \text{...(i)}$$

By bending equation
$$\frac{\sigma}{y} = \frac{E}{R}$$

$$\Rightarrow \quad \frac{1}{R} = \frac{\sigma}{Ey}$$

Putting value of $\frac{1}{R}$ into (i), we get

$$\delta = \frac{\sigma L^2}{8yE} \quad \text{...(ii)}$$

For δ_{max},
$$y_{max} = \frac{t}{2} \text{ and } \sigma_{max} = \frac{3\ Wl}{2\ nbt^2}$$

From (ii), we get

$$\delta_{max} = \frac{\sigma_{max} L^2}{8 \times \frac{t}{2} \times E} = \frac{\frac{3\ Wl}{2\ nbt^2} \times L^2}{4 \times t \times E}$$

$$\therefore \quad \boxed{\delta_{max} = \frac{3\ WL^3}{8\ nbt^3 E}}$$

Though Bending moment and shear force both are developed on the section, but effect of Bending Moment is more than the shear force. Hence it is called **bending spring**.

Example 12.4 A leaf spring is made of plates 50 mm wide and 8 mm thick. The spring has a span of 700 mm. Determine the number of plates required to carry a central load of 45 kN. The maximum allowable stress in the plate is 200 N/mm². What is the maximum deflection under this load.

Solution:

$$L = 700 \text{ mm}$$

$$W = 45 \text{ kN} = 45 \times 10^3 \text{ N}$$

$$t = 8 \text{ mm}$$

$$\sigma_{max} \ngtr 200 \text{ N/mm}^2$$

We know,

$$\sigma_{max} = \frac{3}{2}\frac{WL}{nbt^2}$$

$$200 = \frac{3}{2} \times \frac{45 \times 10^3 \times 700}{n \times 50 \times 8^2}$$

$\therefore$

$$n = 74 \text{ plates}$$

Maximum deflection under the central load,

$$\delta_{max} = \frac{3}{8}\frac{WL^3}{nbt^3E} = \frac{3}{8}\left(\frac{45 \times 10^3 \times 700^3}{74 \times 50 \times 8^3 \times 200 \times 10^3}\right) = 15.27 \text{ mm}$$